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Reducibility in algebraic hyperstructures

PHD THESIS

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Reducibilnost u algebarskim hiperstrukturama

DOKTORSKA DISERTACIJA

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PODACI O DOKTORSKOJ DISERTACIJI

Naziv doktorskih studija: Matematika, Prirodno-matematički fakultet, Univerzitet Crne Gore

Naslov disertacije: Reducibilnost u algebarskim hiperstrukturama

Rezime: Cilj ove disertacije je ekstenzija koncepta reducibilnosti u hipergrupama na fazi slučaj, definisanjem fazi reducibilne hipergrupe tj. klasične hipergrupe obogaćene sa fazi skupom koja je reducibilna. Osim toga, namjeravamo da proširimo koncept reducibilnosti na hiperprstene, takođe, u klasičnom slučaju. Koristeći ove koncepte, naš cilj je da proučavamo svojstvo reducibilnosti (u klasičnom i fazi slučaju) u algebarskim hiperstrukturama, posebno u hipergrupama i hiperprstenima.

Ključne riječi: Reducibilnost, hiperstrukture, relacije ekvivalencije, fundamentalne relacije, fazi reducibilnost

Naučna oblast: Algebra

Uža naučna oblast: Hiperkompoziciona algebra

UDK broj: 519.21

INFORMATION ON THE PHD THESIS

Name of the doctoral program: Mathematics, Faculty of Sciences and Mathematics, University of Montenegro

Thesis title: Reducibility in algebraic hyperstructures

Summary: The goal of this dissertation is to extend the concept of the reducibility in a hypergroup to the fuzzy case, by defining a fuzzy reduced hypergroup, i.e., a crisp hypergroup endowed with a fuzzy set which is reduced. Moreover, we intend to extend the concept of reducibility to the hyperrings, also, in the crisp case. Using these concepts, we aim to study the reducibility and the fuzzy reducibility in algebraic hyperstructures, especially in hypergroups and hyperrings.

Key words: Reducibility, hyperstructures, equivalence relations, fundamental relations, fuzzy reducibility.

Scientific field: Algebra

Scientific subfield: Hypercompositional algebra

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Introduction

The theory of hypercompositional structures (called also the theory of hyperstructures) was introduced in 1934 by F. Marty, when he gave the definition of a hypergroup and presented some of its properties and applications to algebraic functions, rational fractions and non-commutative groups. Hyperstructures represent an independent line of research, but they are also a tool of investigation in many other fields like: Geometry, Graphs and Hypergraphs, Topology, Cryptography, Code Theory, Automata Theory, Probability, Theory of Fuzzy Sets. We may say that the algebraic hypergroups are the most natural generalization of the classical groups: the binary operation of groups is extended to a binary multivalued operation, called hyperoperation or hyperproduct, that associates with any couple of elements of a given set, a non-empty subset of it. In 1934 F. Marty gave the first example of a hypergroup, which was the motivation for introducing this concept. The quotient structure G/H, where G is a group and H is a subgroup of it is not a group, but a hypergroup. In a special case, when H is a normal subgroup, the corresponding quotient becomes a group, which is again a hypergroup. Analogously to the notion of a hypergroup, the other generalizations of algebraic structures have been arised subsequently. The hyperrings are natural generalizations of rings, where one operation in the ring becomes a hyperoperation. Hyperfield is a hyperstructure which generalizes the notion of a field. Similarly, the notions of hyperlattices, hypermodules, etc are introduced. Besides, in [69], T. Vougiouklis introduced the new class of hyperstructures, so called H_v- structures. We may say that H_v- structures generalize the well-known algebraic structures, where the associative and distributive laws are replaced with their weak versions. These algebraic hyperstructures are the subject of interest for many researchers nowadays. For a detailed historical development of algebraic hyperstructures we refer to [32].

The connections between fuzzy sets and algebraic structures were mostly considered by Iranian mathematicians, where they observed that the composition of the elements from H does not give a subset, but a fuzzy set on H. Unlike in an ordinary set, where we have exactly two possibilities: an element belongs to the set, or it does not belong to the set, in the fuzzy set every element has a certain degree of membership. Fuzzy sets were introduced by Zadeh in [71], where he introduced the concept of fuzzy set regarding it as the extension of the notion of the set. In the fuzzy set theory, the membership function takes the values from the segment $[0, 1]$, while in the classical set theory the characteristic function can take only two values, 0 and 1. Actually, the fuzzy set is an ordered pair containing the subset of universe set, and the membership function which maps elements to the segment $[0, 1]$. However, we usually use term fuzzy set when we refer to the membership function.

The fuzzy sets and algebraic structures have been firstly connected in 1971, when A. Rosenfeld gave the definition of fuzzy subgroup of a group. After twenty-eight years, B. Davvaz extended this definition, introducing the fuzzy subhypergroup of a (crisp) hypergroup, which is a fuzzy algebraic hyperstructure. The study of the fuzzy hyperstructures started only a few years ago, with a paper about fuzzy hypergroups. Then, in 2009, V. Leoreanu-Fotea and B. Davvaz introduced the notions of fuzzy hyperrings and fuzzy hypermodules. Very important connection between fuzzy sets and hypergroups was established by Corsini, in [16], by defining a hyperoperation as a mean of fuzzy subsets, obtaining a join space. The other connection, which is very significant for our research is established via the grade fuzzy set $\widetilde{\mu}$, introduced also by Corsini [16]. Besides, the grade fuzzy set is used for the definition of a fuzzy grade, which represents the number of non-isomorphic join spaces and fuzzy sets associated with a given hypergroupoid. The theory of hypergroups associated with fuzzy sets represents a new research direction which preoccupies researchers in the last two decades. Untill now one distinguishes three principal approaches: the study of new crisp hyperoperations obtained by means of fuzzy sets; the study of fuzzy subhypergroups (fuzzy sets whose level sets are crisp hypergroups); the fuzzy hypergroups, i.e., structures endowed with fuzzy hyperoperations. The overview of this theory can be found in the monograph "Fuzzy algebraic hyperstructures: an introduction" written by Davvaz and Cristea [29].

One of the most important concepts in hyperstructure theory are certain relations, being called fundamental relations. These equivalences play a crucial role in obtaining quotient structures. We can divide fundamental relations into two groups: the first group is contained of the relations α, β, γ which are defined on proper hyperstructures such that the obtained quotient structures (hyperstructures modulo relation) are classical algebraic structures. The relation α is defined on a hyperring, while the other two relations (β, γ) are defined on a semihypergroup. Consequently, the resulting quotient structures are ring and semigroup. Moreover, semihypergroup modulo relation γ gives a commutative semigroup.

The fundamental relations are the smallest equivalence relations such that the quotient structures defined on the support set of a hyperstructure becomes a classical structures. These relations represent the link between the classical algebraic structures and algebraic hyperstructures, and besides, it could be noted that the classical algebraic structures impersonate special cases of algebraic hyperstructures. The second group of fundamental relations consists relations introduced by Jantosciak in [42]. He noticed that sometimes a hyperproduct on given set does not make a distinction between a pair of elements of the set. In other words, the elements play exchangeable roles with respect to the hyperoperation. It inspired Jantosciak to define three relation with the aim to identify the elements with the same behaviour.

[42] Two elements x, y in a hypergroup $(H, \circ)$ are called: operationally equivalent, if their hyperproducts with all elements in H are the same: $x \circ a = y \circ a$, and $a \circ x = a \circ y$, for any a in H; inseparable, if x belongs to the same hyperproducts $a \circ b$ as y, for all a, b in H; essentially indistinguishable, if they are operationally equivalent and inseparable.

With the help of these three relations, Jantosciak introduced the concept of reducibility. He defined a reduced hypergroup as a hypergroup where the equivalence class of each element with respect to the essentially indistinguishable relation is a singleton [42]. Moreover, he proved that the quotient hypergroup obtained by factorizing a hypergroup modulo the essential indistinguishable relation always gives a reduced hypergroup, which he called a reduced form. Motivated by this property, the same author proposed that the study of reducibility can be splited in two directions: the study of reduced hypergroups, and the study of all hypergroups having the same reduced form [42] . The study of the reducibility will be also developed in the PhD thesis to the fuzzy case in one direction. We will study indistinguishability between the images of the elements of a classical hypergroup through a fuzzy set. The second direction is studying the indistinguishability between the elements of the fuzzy hypergroup. In particular, we introduce the notion of reduced fuzzy hypergroup, which is a fuzzy hypergroup which is reduced, and the notion of fuzzy reduced hypergroup, which is a hypergroup endowed with a fuzzy set which is reduced. In order to define the concept of a fuzzy reduced hypergroup, we introduce equivalences: fuzzy operation equivalence, fuzzy inseparability and fuzzy essential indistinguishability. Further, a fuzzy reduced hypergroup is defined as a hypergroup where the equivalence class of each element with respect to the fuzzy essential indistinguishability is a singleton [22].

In the **preliminary chapter** we present basic definitions and notions related to the hypergroups, hyperrings and fuzzy sets. In the first part of the chapter we give the definition of a hypergroup, after which we define all particular types of hypergroups which are investigated further in the thesis. Therefter, we give the definition of a fundamental relation, and introduce the relations β^* and γ^*, which provide that factorizing a hypergroup (semihypergroup) by them gives a group (semigroup). Further in the chapter we define all three types of hyperrings. We recall first the hyperring containing an additive hyperoperation and a multiplicative operation, afterwards we present the hyperring with an additive operation and a multiplicative hyperoperation, and at the end we give the definition of a general hyperring, where, both, addition and multiplication are hyperoperations. We present important classes of hyperrings in order to study their reducibility later in the thesis. At the end of the chapter we illustrate the definition of the fuzzy set and explain in detail its connection with algebraic hyperstructures. We explain here the well known fuzzy set $\widetilde{\mu}$, which is used in our study of fuzzy reducibility. Also, we describe a procedure of construction of the sequence of join spaces and fuzzy sets associated with a given hypergroupoid. In the same section we recall the definition of the fuzzy hyperoperation and fuzzy hypersemigroup.

The **second chapter** deals with the reducibility property in hypergroups. First, we present the motivation and expose early ideas related to this concept, as it is presented in the paper of Jantosciak [42], which was the major inspiration for the thesis. In the first part of the chapter, we present some results related to the reducibility in hypergroups associated with binary relations [23]. Then we focus on the study of reducibility for several types of hypergroups. In this chapter we present results which are the subject of the article *Fuzzy reduced hypergroup*, published in *Mathematics*, 2020. by Kankaraš and Cristea [45], and the article *Reducibility in Corsini hypergroups*, by Kankaraš [44]. Then we focus on the study of reducibility for several types of hypergroups. In this chapter we present results which are the subject of the article *Fuzzy reduced hypergroup*, published in *Mathematics*, 2020. by Kankaraš and Cristea, and the article *Reducibility in Corsini hypergroups*, by Kankaraš in *Analele Stiintifice Universitatii Ovidius Constanta, Seria Matematica* in 2021. We prove that any canonical hypergroup is reduced, and as the consequence, we get that any hypergroup contained of partial scalar identities (or i.p.s hypergroup) is reduced, too. The properties of i.p.s. hypergroups presented in this chapter are important for the further study of their fuzzy reducibilility. Further in the chapter we study reducibility for some particular classes of cyclic hypergroups, and we show that their reducibility depends on many conditions. Also, we study reducibility for a very important class of hypergroups, called complete hypergroups, and conclude that any proper complete hypergroup is not reduced. We later use this result for the study of reducibility in complete hyperrings. In the last section of Chapter 2 we give a neccesary and sufficient condition for the Corsini hypergroup to be reduced. As a consequence of this statement, we get that the well-known B-hypergroup, which is a special case of the Corsini hypergroup, is a reduced hypergroup. Also, we determine whether the direct products of hypergroups containing Corsini hypergroups, are reduced or not.

The **third chapter** deals with the fuzzy reducibility in hypergroups, i.e., it contains the study of reducibility in crisp hypergroups endowed with a fuzzy set. In the thesis, the fuzzy reducibility is studied with respect to the grade fuzzy set $\widetilde{\mu}$. In the first part of the chapter, we are introduced to the concept of fuzzy reducibility, which represents the one direction how the reducibility concept can be extended to the fuzzy case. Therefter, we investigate the fuzzy reducibility for several types of hypergroups. The chapter contains results published in the articles *Fuzzy reduced hypergroups* and *Reducibility in Corsini hypergroups*. We prove that any total hypergroup is not reduced, neither fuzzy-reduced. Also, we prove that any proper complete hypergroup is not fuzzy reduced, same as it is the hypergroup with partial scalar identities. Later on, we examine the reducibilility and the fuzzy reducibility for a specific type of non-complete $1-$ hypergroups defined by Corsini and Cristea in [17], and we prove that it is not reduced, nor fuzzy reduced. In the last section we prove that Corsini hypergroup is not fuzzy reduced with respect to the grade fuzzy set $\widetilde{\mu}$. At the end of the section, we consider the direct product of Corsini hypergroups and prove that the resulting productional hypergroup is not fuzzy reduced. The chapter concludes with a brief review of reduced fuzzy hypergroups. This is the second direction of the fuzzyfication of the reducibility concept, which will be the subject of our research in the future.

The reducibility in hyperrings is the topic of the **fourth chapter**. At the beginning of the chapter we introduce new equivalence relations and extend the reducibility concept to the hyperrings. In this case, we introduce equivalence relations with respect to the both, additive and multiplicative hyperoperation. We determine how the reducibility in hyperrings depends on the reducibility in hypergroupoids of which it is composed. Further, we examine the reducibility for the specific types of general hyperrings. In particular, we prove that any complete hyperring is not reduced. We determine conditions such that the $(H, R)-$ hyperring is reduced. Also, we present some properties of the reducibility in some particular types of hyperrings, as H_v- rings with $P-$ hyperoperations, hyperrings of formal series and others.

The **last chapter** contains some new research ideas concerning this study. Some aims of our further research are related to the study of the reducibility in fuzzy hyperstructures, especially in fuzzy hypergroups. Also, we intend to extend the fuzzy reducibility concept for the hyperrings and investigate the fuzzy reducibiity for certain types of general hyperrings.

Podgorica, March 2022. Milica Kankaraš

Izvod iz teze

Teoriju hiperkompozicionalnih struktura (koja se još naziva i teorijom hiperstruktura) uveo je 1934. francuski matematičar F. Marty, kada je definisao hipergrupu i prikazao neka njena svojstva i primjene u oblastima algebarskih funkcija i ne-komutativnih grupa. Hiperstrukture predstavljaju nezavisnu oblast istraživanja, a takođe mogu da služe i kao "alat" za istraživanje u drugim oblastima kao što su: Geometrija, Grafovi i Hipergrafovi, Topologija, Kriptografija, Teorija kodiranja, Teorija automata, Vjerovatnoća, Teorija fazi skupova. Možemo reći da su algebarske hipergrupe prirodno uopštenje klasičnih grupa: binarna operacija grupe se proširuje na binarnu multivrijednosnu operaciju, nazvanu hiperoperacijom ili hiperproizvodom, koja svakom paru elemenata zadatog skupa pridružuje njegov neprazni podskup. F. Marty je 1934. godine dao prvi primjer hipegrupe, što je ujedno bila motivacija za uvođenje ovog koncepta. Ako je G grupa, a H njena podgrupa tada količnička (faktor) struktura H/G u opštem slučaju nije grupa, nego hipergrupa. U specijalnom slučaju, kada je H normalna podgrupa, odgovarajuća količnička struktura je grupa, a grupa je ujedno i hipergrupa.

Za neprazan skup H, neka je $\mathcal{P}^*(H)$ familija nepraznih podskupova skupa H. Binarna hiperoperacija, koju još nazivamo i hiperproizvodom je preslikavanje $\circ : H \times H \to \mathcal{P}^*(H)$, a uređeni par $(H, \circ)$ se naziva hipergrupoidom. Važno je naglasiti da je u hipergrupoidu hiperproizvod $x \circ y$ dva proizvoljna elementa x i y iz H neprazan podskup skupa H, dok je u klasičnim algebarskim strukturama, rezultat binarne operacije između dva elementa samo jedan element inicijalnog skupa (koji se još naziva i nosač). Ako je hiperoperacija asocijativna, tj. važi $(a \circ b) \circ c = a \circ (b \circ c)$, za sve elemente $a, b, c \in H$, tada se hiperkompozicionalna struktura $(H, \circ)$ naziva semihipergrupom. Semihipergrupa postaje hipergrupa ako važi i reproducibilnost : $x \circ H = H \circ x = H$ za sve $x \in H$.

Hiperoperacija $\circ$ se proširuje na neprazne skupove A, B skupa H i za $x \in H$, važi [32]:

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b \quad A \circ x = A \circ \{x\} \quad x \circ B = \{x\} \circ B.$$

Dakle, jednakost $(a \circ b) \circ c = a \circ (b \circ c)$ implicira da

$$\bigcup_{u \in a \circ b} u \circ c = \bigcup_{v \in b \circ c} a \circ v.$$

Analogno pojmu hipergrupe, uskoro su se pojavile i ostale generalizacije algebarskih struktura. Hiperprsteni su generalizacije struktura prstena, gdje se jedna od operacija u prstenu zamijeni hiperoperacijom. Hiperpolje uopštava pojam polja. Slično su uvedeni i pojmovi hiperrešetki, hipermodula. Osim toga, T. Vougiouklis je u [69], uveo novu klasu hiperstruktura, takozvanih H_v- struktura. H_v- strukture su generalizacije poznatih algebarskih struktura, gdje su asocijativni i distributivni zakoni zamijenjeni njihovim "slabijim" verzijama. Ove algebarske strukture su predmet interesovanja velikog broja istraživača danas. Za detaljan istorijski razvoj teorije algebarskih hiperstruktura, čitaocima preporučujemo knjigu [32].

Vezama između fazi skupova i algebarskih struktura su se pretežno bavili iranski matematičari, koji su posmatrali kompozicije elemenata iz skupa H koje kao rezultat ne daju podskup skupa H, već fazi skup na H. Za razliku od klasičnog skupa, gdje element pripada ili ne pripada skupu, fazi skup dozvoljava da element ima određeni stepen pripadnosti skupu. Pojam fazi skupa je uveo Zadeh, u [71], gdje je definisao koncept fazi skupa kao proširenje pojma skupa. U teoriji fazi skupova, funkcija pripadnosti uzima vrijednosti sa segmenta $[0, 1]$, dok u klasičnoj teoriji skupova karakteristična funkcija uzima samo vrijednosti 0 ili 1. Preciznije, fazi skup se definiše kao uređeni par koji sadrži podskup univerzalnog skupa, i funkciju pripadnosti koja preslikava element iz tog skupa na segment $[0, 1]$. Međutim, često upotrebljavamo termin fazi skup kada govorimo o funkciji pripadnosti.

Definicija 0.1. [32] *Neka je X skup. Fazi podskup A skupa X se karakteriše funkcijom pripadnosti $\mu_A : X \to [0, 1]$ koja svakoj tački $x \in X$ pridružuje ocjenu ili stepen pripadnosti $\mu_A(x) \in [0, 1]$.*

Prve veze izmedju fazi skupova i algebarskih struktura uspostavio je A. Rosenfeld 1971. godine, kada je definisao pojam fazi podgrupe grupe. 28 godina kasnije, B. Davvaz je proširio ovu definiciju na slučaj algebarskih hiperstruktura, uvodeći koncept fazi podhipergrupe (obične) hipergrupe. Proučavanje fazi hiperstruktura je započelo samo par godina prije, člankom o fazi hipergrupama. 2009. godine V. Leoreanu-Fotea i B. Davvaz uvode pojmove fazi hiperprstena i fazi hipermodula. Veoma značajnu vezu između fazi skupova i hipergrupa uspostavio je Corsini, u [16], gdje je definisao hiperoperaciju kao sredinu fazi podskupova, dobivši tako *pridruženi prostor.* Druga konekcija, koja je od velikog značaja za naše istraživanje, uspostavljena je uz pomoć *grade fazi skupa (grade fuzzy set)* $\widetilde{\mu}$, kog je takođe uveo Corsini [16]. Grade fazi skup se koristi za definisanje *fazi grade-a*, koji predstavlja broj neizomorfnih pridruženih prostora i fazi skupova povezanih sa zadatim hipergrupoidom. Teorija hipergrupa povezanih sa fazi skupovima predstavlja novi pravac u istraživanju u teoriji hiperstruktura koji je doživio ekspanziju posljednjih 20 godina. Razlikujemo tri osnovna pristupa u ovom istraživanju: Izučavanje "običnih" hiperoperacija dobijenih pomoću sredina fazi skupova; Izučavanje fazi podhipergrupa (fazi skupovi čiji su nivo skupovi "obične" hipergrupe); Izučavanje fazi hipergrupa, tj. struktura obogaćenih sa fazi hiperoperacijama. Monografija "Fuzzy algebraic hyperstructures: an introduction" čiji su autori Davvaz i Cristea [29], sadrži pregled ove teorije.

U teoriji hiperstruktura određene ekvivalencije igraju ključnu ulogu u dobijanju količničkih struktura, a te relacije nazivamo fundamentalnim relacijama. Fundamentalne relacije mogu da se podijele u dvije grupe: prva grupa sadrži relacije α, β, γ definisane na hiperprstenu (prva) i na semihipergrupi (druge dvije), takve da dobijena količnička struktura predstavlja prsten, semigrupu i komutativnu semigrupu (polugrupu), respektivno. Fundamentalne relacije su najmanje relacije ekvivalencije takve da količničke strukture koje dobijamo faktorisanjem hiperstruktura po ovim relacijama postaju klasične strukture. Ove relacije predstavljaju "most" između klasičnih algebarskih struktura i hiperstruktura, a osim toga, primijećujemo da sada klasične strukture možemo da posmatramo kao specijalne slučajeve algebarskih hiperstruktura.

Da bismo uveli preciznu definiciju fundamentalnih relacija potrebno je definisati regularne relacije, kao i jako (strogo) regularne relacije.

Definicija 0.2. [13] *Neka je $(H, \circ)$ hipergrupoid, $a, b \in H$ i ρ je relacija ekvivalencije na H. Tada je relacija ρ lijevo regularna ako:*

$$\begin{array}{c} a\rho b \Rightarrow (\forall u \in H, \forall x \in u \circ a, \exists y \in u \circ b: \quad x\rho y \\ i \\ \forall u \in H, \forall y' \in u \circ b, \exists x' \in u \circ a: \quad x'\rho y') \end{array} \tag{1}$$

Relacija ρ je desno regularna ako:

$$\begin{array}{c} a\rho b \Rightarrow (\forall u \in H, \forall x \in a \circ u, \exists y \in b \circ u: \quad x\rho y \\ i \\ \forall u \in H, \forall y' \in b \circ u, \exists x' \in a \circ u: \quad x'\rho y') \end{array} \tag{2}$$

Relacija ρ je regularna ako je lijevo i desno regularna.

Definicija 0.3. [13] *Neka je* $(H, \circ)$ *hipergrupoid,* $a, b \in H$ *i* ρ *je relacija ekvivalencije na* H*. Tada je relacija* ρ *jako lijevo regularna ako:*

$$a\rho b \Rightarrow \forall u \in H, \forall x \in u \circ a, \forall y \in u \circ b : \quad x\rho y.$$

Relacija ρ *je jako desno regularna ako:*

$$a\rho b \Rightarrow \forall u \in H, \forall x \in a \circ u, \forall y \in b \circ u : \quad x\rho y.$$

Relacija ρ *je jako regularna ako je jako lijevo i jako desno regularna.*

Tvrđenje 0.1. [13] *Ukoliko je* $(H, \circ)$ *hipergrupa i* R *je relacije ekvivalencije na* H*, tada je* R *regularna ako i samo oko je* $(H/R, \otimes)$ *hipergrupa, gdje je:* $\bar{x} \otimes \bar{y} = \{\bar{z} : z \in x \circ y\}$.

Tvrđenje 0.2. [32] *Ako je* $(H, \circ)$ *hipergrupa, a* R *relacija ekvivalencije na* H*, tada je* R *jako regularna ako i samo ako je* $(H/R, \otimes)$ *grupa.*

Jedna od najpoznatijih i najznačajnih fundamentalnih relacija u teoriji hiperstruktura je relacija β.

Definicija 0.4. [46] *Neka je* $(H, \circ)$ *semihipergrupa i* $n \geq 1, n \in \mathbb{N}$*. Definišimo relaciju* β_n *na sljedeći način:*

$$x\beta_n y \quad \text{ako postoje} \quad a_1, a_2, \ldots, a_n \quad \text{tako da} \quad \{x, y\} \subseteq \prod_{i=1}^{n} a_i$$

i neka

$$\beta = \cup_{n\geq 1}\beta_n, \quad \text{gdje je} \quad \beta_1 = \{(x, x) | x \in H\}$$

dijagonalna relacija na H*.*

Relacija β je refleksivna i simetrična. Označimo sa β^* tranzitivno zatvorenje relacije β.

Teorema 0.1. [46] β^* *je najmanja jako regularna relacija ekvivalencije na* H *u smislu inkluzije.*

Teorema 0.2. [46] *Neka je* $(H, \circ)$ *semihipergrupa (hipergrupa), tada je relacija* β^* *najmanja relacija ekvivalencije takva da je količnički skup* H/β^* *semigrupa (grupa).*

Relacija β^* se naziva *fundamentalnom relacijom* na H, a količnički skup H/β^* se naziva *fundamentalnom semigrupom (grupom)*. Važno je naglasiti da se u hipergrupi fundamentalna relacija β poklapa sa relacijom β^* [32]. Dakle, količnički skup dobijen faktorisanjem hipergrupe (sa odgovarajućom operacijom) po relaciji β je grupa.

U poglavlju **Preliminaries** se mogu naći definicije drugih značajnih fundamentalnih relacija.

Drugu grupa relacija koje se takođe nazivaju fundamentalnim čine relacije koje je uveo Jantosciak [42] da bi definisao pojam reducibilne hipergrupe.

Jantosciak je primijetio da hiperproizvod na zadatom skupu nekada ne pravi razliku između para elemenata u skupu, tj., elementi nekada igraju istu ulogu u odnosu na zadatu hiperoperaciju. Jantosciak je, motivisan primiječenim ponašanjem elemenata definisao određene ekvivalencije u cilju identifikovanja elemenata sa istim ponašanjem.

[42] Dva elementa x, y u hipergrupi $(H, \circ)$ su: *operaciono ekvivalentna*, ako su njihovi hiperproizvodi sa svim elementima u H isti: $x \circ a = y \circ a$, i $a \circ x = a \circ y$, za sve a u H; *nerazdvojiva*, ako x pripada istim hiperproizvodima $a \circ b$ kao y, za sve a, b in H; *esencijalno nerazlikujuća*, ako su operaciono ekvivalentni i nerazdvojivi.

Definicija 0.5. [42] *Reducibilna hipergrupa je hipergrupa u kojoj je klasa ekvivalencije svakog elementa u odnosu na relaciju esencijalno nerazlikujući $\sim_e$ jednoelementni skup, tj., za sve elemente $x \in H$, važi $\hat{x}_e = \{x\}$.*

Osim gore navedene definicije, isti autor je dokazao da je količnička hipergrupa, dobijena faktorisanjem hipergrupe po relaciji esencijalno nerazlikujući uvijek reducibilna i nazvao ju je reducubilna formom. Motivisan ovim svojstvom, pomenuti autor je predložio da istraživanje reducibilnosti može dalje da se razvija u dva smjera: izučavanje reducibilnih grupa i izučavanje svih hipergrupa koje imaju istu reducibilnu formu.

U daljem tekstu dajemo primjer reducibilne hipergrupe.

Primjer 0.6. Neka je $(H, \circ)$ hipergrupa, gdje je hiperoperacija " $\circ$ " definisana sa sljedećom tabelom:

$\circ$	a	b	c	d
a	a	a	a, b, c	a, b, d
b	a	a	a, b, c	a, b, d
c	a, b, c	a, b, c	a, b, c	c, d
d	a, b, d	a, b, d	c, d	a, b, d

(3)

Lako je primijetiti da $a \sim_o b$, jer su vrste (i kolone) koje odgovaraju elementima a i b potpuno iste, otuda: $\hat{a}_o = \hat{b}_o = \{a, b\}$, dok je $\hat{c}_o = \{c\}$ i $\hat{d}_o = \{d\}$. S druge strane, klasa ekvivalencije svakog elementa skupa H u odnosu na relaciju $\sim_i$ je jednoelementni skup, kao i u odnosu na relaciju $\sim_e$. Posljedično, hipergrupa $(H, \circ)$ je reducibilna.

Izučavanje redicibilnosti će u mojoj disertaciji biti prošireno na fazi slučaj u jednom pravcu. Posmatraćemo esencijalno nerazlikovanje između slika elemenata u klasičnoj hipergrupoj "kroz" fazi skup. Drugi pravac je izučavanje esencijalnog nerazlikovanja između elemenata fazi hipergupe. U cilju izučavanja reducibilnosti proširene na "fazi slučaj", uvodimo pojam *reducibilne fazi hipergrupe*, tj. fazi hipergrupe koja je reducibilna, kao i pojam *fazi reducibilne hipergrupe*, tj. hipergrupe obogaćene sa fazi skupom (na kojoj je defisan fazi skup) koja je reducibilna. Da bismo definisali koncept fazi reducibilne hipegrupe, uvodimo nove ekvivalencije po ugledu na one koje je definisao Jantosciak: *fazi operaciona ekvivalentnost,fazi nerazlikovanje* i *fazi esencijalno nerazlikovanje.*

Definicija 0.7. [22] *U hipergrupi $(H, \circ)$ na kojoj je zadat fazi skup μ, definišemo sljedeće ekvivalencije:*

1. *x i y su fazi operaciono ekvivalentni i pišemo $x \sim_{fo} y$ ako, za sve $a \in H$, $\mu(x \circ a) = \mu(y \circ a)$ i $\mu(a \circ x) = \mu(a \circ y)$;*
2. *x i y su fazi nerazdvojivi i pišemo $x \sim_{fi} y$ ako $\mu(x) \in \mu(a \circ b) \Longleftrightarrow \mu(y) \in \mu(a \circ b)$, za $a, b \in H$;*
3. *x i y su fazi esencijalno nerazlikujući i pišemo $x \sim_{fe} y$, ako su oni fazi operaciono ekvivalentni i fazi nerazdvojivi.*

[22] Hipergrupa $(H, \circ)$ je fazi reducibilna hipergrupa ako je klasa ekvivalencije svakog elementa u odnosu na relaciju fazi esencijalno nerazlikovanje jednoelementni skup.

Važnu klasa hiperstruktura koju ćemo posmatrati u disertaciji čine strukture hiperprstena. Hiperprsteni su hiperkompozicionalne strukture na kom su zadate dvije (hiper) operacije (tako da nisu obje operacije), sa sličnim svojstvima koje imaju operacije u prstenu. Postoje različiti tipovi hiperprstena u zavisnosti od toga kako su zadati aditivni i multiplikativni dio, tj., da li su oni definisani kao operacije ili hiperoperacije. Hiperprsten može da se definiše pomoću dvije hiperoperacije, ili sa jednom operacijom i jednom hiperoperacijom. Razlikujemo tri tipa hiperprstena: aditivni, multiplikativni i generalni. Aditivni hiperprsten je hiperstruktura na kojoj su zadate aditivna hiperoperacija i multiplikativna operacija, gdje je multiplikativna operacija distributivna u odnosu na aditivnu hiperoperaciju. Najpoznatiji aditivni hiperprsten je definisao Krasner 1983. godine [47]. Multiplikativni hiperprsten je hiperprsten na kome su zadate aditivna operacija i multiplikativna hiperoperacija, gdje je multiplikativna hiperoperacija distributivna u odnosu na aditivnu operaciju. Najširu klasu hiperprstena čine generalni hiperprsteni. Generalni hiperprsteni su hiperstrukture na kojima su zadate dvije hiperoperacije, povezane distributivnim svojstvom.

Definicija 0.8. [52] *Hiperkompozicionalna struktura* $(R, \oplus, \odot)$ *je hiperringoid ako*

1. $(R, \oplus)$ *je hipergrupa.*

2. $(R, \odot)$ *je semigrupa.*

3. *Operacija "*$\odot$*" je distributivna s obje strane u odnosu na hiperoperaciju "*$\oplus$*."*

Ako su i sabiranje i množenje hiperoperacije, tada hiperringoid postaje generalni hiperprsten.

Definicija 0.9. [67] *Uređena trojka* $(R, \oplus, \odot)$ *je generalni hiperprsten ako:*

1. $(R, \oplus)$ *je hipergrupa.*

2. $(R, \odot)$ *je semihipergrupa.*

3. *Množenje* $\odot$ *je distributivno u odnosu na* $\oplus$, *tj., za sve* $a, b, c \in R$ $\quad a \odot (b \oplus c) = a \odot b \oplus a \odot c$ *i* $(a \oplus b) \odot c = a \odot c \oplus b \odot c$.

U poglavlju **Preliminaries** prezentujemo osnovne definicije i pojmove povezane sa hipergrupama, hiperprstenima i fazi skupovima. U prvom dijelu ovog poglavlja dajemo definiciju hipergrupe, nakon čega definišemo sve tipove hipergrupa koje su predmet istraživanja dalje u tezi. Nakon toga, uvodimo definiciju fundamentalne relacije i uvodimo relacije β^* i γ^*, koje omogućavaju da faktorisanje hipergrupe (semihipergrupe) po datim relacijama daje grupu (semigrupu). Dalje u ovom poglavlju definišemo sva tri tipa hiperprstena. Na početku se podsjećamo definicije hiperprstena koji sadrži aditivnu hiperoperaciju i multiplikativnu operaciju, a nakon toga predstavljamo prsten sa aditivnom operacijom i multiplikativnom hiperoperacijom, a zatim uvodimo definiciju generalnog hiperprstena, gdje su i sabiranje i množenje hiperoperacije. Osim toga, predstavljamo važne klase hiperprstena s ciljem da izučavamo njihovu reducibilnost kasnije u tezi. Na kraju poglavlja uvodimo definiciju fazi skupa i detaljno objašnjavamo njegovu vezu sa algebarskim hiperstrukturama. Takođe, u ovom poglavlju izučavamo bitna svojstva poznatog fazi skupa $\widetilde{\mu}$ (*grade fazi skup*), koji koristimo dalje u proučavanju fazi redicibilnosti. Takođe, opisujemo proceduru konstrukcije niza pridruženih prostora i fazi skupova povezanih sa zadatim hipergrupoidom. U istoj sekciji uvodimo definicije fazi hiperoperacije i fazi hipersemigrupe.

Drugo poglavlje se bavi sa ispitivanjem reducibilnosti u hipergrupama. U prvom dijelu ovog poglavlja izlažemo motivaciju i rane ideje koje su povezane s ovim konceptom, kao što je prikazano u članku [42], koji je predstavljao glavnu inspiraciju za pisanje ove teze. U prvom dijelu poglavlja, prezentujemo neke rezultate o reducibilnosti hipergrupa povezanih sa binarnim relacijama [23]. Zatim se fokusiramo na istraživanje reducibilnosti za više različitih tipova hipergrupa. U ovom poglavlju predstavljamo rezultate koji su objavljeni u člancima *Fuzzy reduced hypergroup*, koji su 2020. godine objavile Kankaraš i Cristea u *Mathematics*, i članka *Reducibility in Corsini hypergroups*, koji je objavila Kankaraš u *Analele Stiintifice Universitatii Ovidius Constanta, Seria Matematica* 2021. godine. Dokazujemo da je proizvoljna kanonska hipergrupa reducibilna, i kao posljedicu dobijamo da je proizvoljna hipergrupa sa parcijalnim skalarnim identitetima (ili i.p.s. hipergrupa) takođe reducibilna. Svojstva i.p.s. hipergrupe prezentovana u ovom poglavlju su značajna za dalje istraživanje fazi reducibilnosti ovih hipergrupa. Dalje u ovom poglavlju izučavamo reducibilnost za određene klase cikličnih hipergrupa, i pokazujemo da njihova reducibilnost zavisi od više uslova. Takođe, izučavamo reducibilnost jako značajne klase hipergrupa, tzv. kompletnih grupa i zaključujemo da svaka prava kompletna hipergupa nije reducibilna. Ovaj rezultat koristimo kasnije pri izučavanju reducibilnosti kompletnih hiperprstena. U posljednjoj sekciji drugog poglavlja dajemo neophodan i dovoljan uslov da Korsinijeva hipergrupa (Corsini hypergroup) bude reducibilna. Kao posljedicu ovog tvrđenja, zaključujemo da je dobro poznata B-hipergrupa, koja je specijalan slučaj Korsinijeve hipegrupe, reducibilna hipergrupa. Takođe, ispitujemo da li su direktni proizvodi Korsinijevih hipergrupa reducibilni.

Treće poglavlje se bavi fazi reducibilnošću u hipergrupama, tj. ispitivanjem reducibilnosti u "običnim hipergrupama" na kojima je zadat fazi skup. U prvom dijelu poglavlja, uvodimo koncept fazi reducibilnosti, koji predstavlja jedan od pravaca kako koncept reducibilnosti može da se proširi na fazi slučaj. Kasnije istražujemo fazi reducibilnost za više klasa hipergrupa. Poglavlje sadrži rezultate objavljene u člancima *Fuzzy reduced hypergroups* i *Reducibility in Corsini hypergroups*. Dokazujemo da proizvoljna totalna hipergrupa nije reducibilna, niti fazi reducibilna. Takođe, dokazujemo da nijedna prava kompletna hipergrupa nije fazi reducibilna, a nakon toga isto pokazujemo i za hipergrupu sa parcijalnim skalarnim identitima (i.p.s. hipergrupa). Nakon toga, ispitujemo reducibilnost i fazi reducibilnost za posebnu klasu ne-kompletnih $1-$ hipergrupa koju su definisali Corsini i Cristea u [17], i pokazujemo da navedena hipergrupa nije reducibilna, niti fazi reducibilna. U posljednjoj sekciji dokazujemo da Korsinijeva hipergrupa nije fazi reducibilna u odnosu na grade fazi skup $\widetilde{\mu}$. Na kraju sekcije, posmatramo direktni proizvod Korsinijevih hipergrupa i pokazujemo da rezultujuća hipergrupa nije fazi reducibilna. Poglavlje završavamo sa kratkim pregledom fazi hipergrupa. Ovo je drugi pravac "fuzifikacije" koncepta reducibilnosti, koji će biti predmet našeg istraživanja u budućnosti.

Reducibilnost u hiperprstenima je tema **četvrtog poglavlja** ove disertacije. Lako je primijetiti da su u semigrupi (grupi), ekvivalencije $\sim_o$ and $\sim_i$ ekvivalentne relaciji jednakosti, što znači da $x \sim_o y \iff x \sim_i y \iff x = y$, pa nam izučavanje reducibilnosti u hiperprstenima sa jednom hiperoperacijom i jednom operacijom nije od značaja. Preciznije, nećemo se baviti izučavanjem reducibilnosti u aditivnom i multiplikativnom prstenu. Dakle, izučavaćemo reducibilnost samo u generalnim hiperprstenima, gdje su i sabiranje i množenje hiperoperacije. Za elemente $x \in R$, označimo sa $\hat{x}_r^{\oplus}$ i $\hat{x}_r^{\oplus}$ njihove klase ekvivalencije u odnosu na hiperoperacije $\oplus$ i $\odot$, respektivno, gdje $r \in \{o, i, e\}$ predstavlja tip ekvivalencije koji razmatramo.

Definicija 0.10. [21] *Kažemo da su dva elementa x i y u hiperprstenu $(R, \oplus, \odot)$ operaciono ekvivalentna, nerazdvojiva ili esencijalno nerazlikujuća ako imaju isto svojstvo u odnosu na obje hiperoperacije, tj.*

1. *$x \sim_o y$ ako $x \oplus a = y \oplus a, a \oplus x = a \oplus y$ i $a \odot x = a \odot y, x \odot a = y \odot a$, za sve $a \in R$.*
2. *$x \sim_i y$ ako $x \in a \oplus b \iff y \in a \oplus b$ i $x \in c \odot d \iff y \in c \odot d$, za sve $a \in R$.*
3. *Osim toga, $x \sim_e y$ ako $x \sim_o y$ i $x \sim_i y$.*

Slično kao u hipergrupama, uvodimo definiciju reducibilnog hiperprstena, koristeći gore definisane relacije.

Definicija 0.11. [21] *Hiperprsten R je reducibilni hiperprsten ako je klasa ekvivalencije svakog elementa $x \in R$ u odnosu na relaciju esencijalno nerazlikujući $\sim_e$ jednoelementni skup, tj., $\hat{x}_e = \{x\}$ za sve $x \in R$.*

Klasa ekvivalencije elementa x in R u odnosu na relaciju esencijalno nerazlikujući $\sim_e$ se dobija kao $\hat{x}_e = \hat{x}_e^{\oplus} \cap \hat{x}_e^{\odot} = (\hat{x}_o^{\oplus} \cap \hat{x}_i^{\oplus}) \cap (\hat{x}_o^{\odot} \cap \hat{x}_i^{\odot})$. Lako se primjećuje da, ako je makar jedan od hipergrupoida $(R, \oplus)$ or $(R, \odot)$ reducibilan, tada je i hiperprsten $(R, \oplus, \odot)$ takođe reducibilan. Obrnuto, ako je $(R, \oplus, \odot)$ reducibilan hiperprsten, tada hipergrupoidi $(R, \oplus)$ i $(R, \odot)$ mogu a i ne moraju da budu reducibilni.

Određujemo kako reducibilnost u hiperprstenima zavisi od reducibilnosti u hipergrupoidima od kojih je sačinjen. Dalje, ispitujemo reducibilnost u specifičnim tipovima generalnih prstena. Posebno, dokazujemo da kompletni hiperprsteni nisu reducibilni. Nakon toga određujemo uslove koji impliciraju reducibilnost $(H, R)-$ hiperprstena. Takođe, prezentujemo neka zanimljiva svojstva reducibilnosti u određenim tipovima hiperprstena, kao što su H_v- prsteni sa $P-$ hiperoperacijama, $\triangle-$ prsteni, hiperprsteni formalnih redova i drugi.

U **Posljednjem poglavlju** su prezentovane nove ideje koje su povezene sa našim istraživanjem. Jedan od ciljeva našeg daljeg istraživanja je izučavanje reducibilnosti u fazi hiperstrukturama, posebno u fazi hipergrupama.

Definition 0.12. [60] *Neka je S neprazan skup. Fazi hiperoperacija na S je preslikavanje $\circ : S \times S \to F(S)$, gdje je $F(S)$ skup svih fazi podskupova skupa S. Struktura $(S, \circ)$ je fazi hipergrupoid.*

Definicija 0.13. [60] *Fazi hipergrupoid $(S, \circ)$ je fazi hipersemigrupa ako za sve $a, b, c \in S$, $(a \circ b) \circ c = a \circ (b \circ c)$ i za svaki fazi podskup μ na S važi*

$$(a \circ \mu)(r) = \begin{cases} \vee_{t \in S}((a \circ t)(r) \wedge \mu(t)), & ako \quad \mu \neq \emptyset \\ 0, inače \end{cases} \tag{4}$$

$$(\mu \circ a)(r) = \begin{cases} \vee_{t \in S}(\mu(t) \wedge (t \circ a)(r), & ako \quad \mu \neq \emptyset \\ 0, inače \end{cases} \tag{5}$$

za sve r u S.

Definicija 0.14. [60] *Fazi hipersemigrupa $(S, \circ)$ je fazi hipergrupa ako je $x \circ S = S \circ x = \chi_S$, za sve x u S, gdje je χ_S karakteristična funkcija skupa $S, tj.$,*

$$\chi_S(x) = \begin{cases} 1, if & x \in S \\ 0, if & x \notin S. \end{cases} \tag{6}$$

Da bi definisali reducibilnu fazi hipergrupu, uvodimo nove relacije ekvivalencije na fazi hipergrupi, tj. na hipergrupi na kojoj je zadata fazi hiperoperacija. Relacije imaju ista imena kao u slučaju "obične" hipergrupe: operaciona ekvivalentnost, nerazdvojivost i esencijalno nerazlikovanje.

[22] Dva elementa x, y u hipergrupi $(H, \circ)$ su:

1. *operaciono ekvivalentna* ili *o-ekvivalentna*, i pišemo $x \sim_o y$, ako $(x \circ a)(r) = (y \circ a)(r)$, i $(a \circ x)(r) = (a \circ y)(r)$, za sve $a, r \in H$;

2. *nerazdvojiva* ili *i-ekvivalentna*, i pišemo $x \sim_i y$, ako za sve $a, b \in H, x \in supp(a \circ b) \iff y \in supp(a \circ b)$, tj., $(a \circ b)(x) \neq 0 \iff (a \circ b)(y) \neq 0$;

3. *esencijalno nerazlikujući* ili *e-ekvivalentni*, i pišemo $x \sim_e y$, ako su operaciono ekvivalentni i nerazlikujući.

Definicija 0.15. [22] $(H, \circ)$ *je reducibilna fazi hipergrupa ako i samo ako za sve* $x \in H$ *važi* $\hat{x}_e = \{x\}$.

U posljednjem poglavlju se nalaze primjeri reducibilnih fazi hipergrupa koji su motivacija za dalji nastavak istraživanja ovog svojstva. Osim toga, namjeravamo da u budućnosti proširimo koncept fazi reducibilnosti na hiperprstene i istražujemo fazi reducibilnost u određenim tipovima generalnih hiperprstena.

Podgorica, 2022. Milica Kankaraš

Abstract

This thesis deals with the reducibility property in algebraic hypercompositional structures. The concept of reducibility was introduced by Jantosciak, when he defined certain equivalences in order to identify elements which have the same role with respect to the hyperoperation. He defined the operational equivalence, inseparability and essential indistinguishability in a hypergroup and called these relations fundamental. Besides, he gave a definion of a reduced hypergroup as a hypergroup where the equivalence class of any element with respect to the relation "essential indistinguishability" is a singleton. Based on the relations defined by Jantosciak, we introduce new equivalence relations on a crisp hypergroup endowed with a fuzzy set and call them the fuzzy operational equivalence, fuzzy inseparability and fuzzy essential indistinguishability, i.e., we extend the reducibility concept to the fuzzy case. We define a fuzzy reduced hypergroup as a hypergroup where every element has a singleton equivalence class with respect to the fuzzy essential indistinguishability. Further more, the extension can go to another direction, which leads to the study of the reducibility in fuzzy hyperstructures, which are hyperstructures endowed with fuzzy hyperoperations. The fuzzy reducibility strictly depends on the given fuzzy set, but in our thesis we will only consider the fuzzy set $\widetilde{\mu}$ defined by Corsini and called grade fuzzy set. In the second part of the thesis, the concept of the reducibility is extended to general hyperrings. The equivalence relations are defined with respect to the both, additive and multiplicative hyperoperations. After presenting some general properties and examples of reduced hyperrings, our study focuses on particular types of hyperrings. They are complete hyperrings, $(H, R)-$ hyperrings, $\triangle-$ hyperrings and the hyperring of formal series. The thesis ends with a conclusive part containing also some ideas of future works.

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Chapter 1

Preliminaries

This chapter gathers together the basic notions and results related to hypergroups and hyperrings. For a detailed overview we referee the readers to the fundamental books [13, 32].

1.1 Hypergroups

For a non-empty set H, we denote by $\mathcal{P}^*(H)$ the family of all non-empty subsets of H. A binary hyperoperation, called also a hyperproduct, is an application $\circ : H \times H \to \mathcal{P}^*(H)$ and the pair $(H, \circ)$ is called a *hypergroupoid*. If hyperoperation $\circ$ mapps $H \times H$ to $\mathcal{P}(H)$, where $\mathcal{P}(H)$ the family of all subsets of H (including the empty one), then pair $(H, \circ)$ is called a *partial hypergroupoid*. It is important to stress that in a hypergroupoid the hyperproduct $x \circ y$ between two arbitrary elements x and y in H is a non-empty subset of H, while in classical algebraic structures, the result of a binary operation between two elements is just one element of the initial set (called the support set). If the associativity is also valid, i.e., $(a \circ b) \circ c = a \circ (b \circ c)$, for all $a, b, c \in H$, then the hypercompositional structure $(H, \circ)$ is a semihypergroup that becomes a hypergroup when also the reproducibility property holds: $x \circ H = H \circ x = H$ for all $x \in H$.

The hyperoperation $\circ$ is extended also to non-empty subsets A, B of H and for $x \in H$, there is [32]

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b \quad A \circ x = A \circ \{x\} \quad x \circ B = \{x\} \circ B.$$

So the associative property $(a \circ b) \circ c = a \circ (b \circ c)$ means that

$$\bigcup_{u \in a \circ b} u \circ c = \bigcup_{v \in b \circ c} a \circ v.$$

If the hyperoperation $\circ$ satisfies just the reproducibility, then $(H, \circ)$ is called a *quasi-hypergroup* [32].

A hypergroupoid $(H, \circ)$ which is both a semihypergroup and a quasihypergroup is called a *hypergroup* [32].

For the representation of a certain finite hypergroup H we often use the *Cayley table*. The Cayley table describes the hyperoperation action on every pair of elements in H. In the following example, we represent the hypergroupoid $(H, \circ)$ with the Cayley table and show that the given hypergroupoid is a hypergroup.

Example 1.1. On the set $H = \{a, b, c, d\}$ define the hyperoperation $\circ$ by the following Cayley table:

$\circ$	a	b	c	d
a	a	a	a, b, c	a, b, d
b	a	a	a, b, c	a, b, d
c	a, b, c	a, b, c	a, b, c	c, d
d	a, b, d	a, b, d	c, d	a, b, d

(1.1)

Let us first check whether the reproduction axiom is valid, i.e., whether the hyperproduct of any element x with the set H gives the whole set H.
The hyperproduct $a \circ H$ is equal to $\bigcup_{x \in H} a \circ x = a \circ a \cup a \circ b \cup a \circ c \cup a \circ d = \{a\} \cup \{a\} \cup \{a, b, c\} \cup \{a, b, d\} = \{a, b, c, d\} = H$. Due to the commutativity (the table is symmetrical about the main diagonal), the hyperproduct $H \circ a$ is equal to $a \circ H$, which means that $a \circ H = H \circ a = H$. Similarly it can be proved that $b \circ H = c \circ H = d \circ H = H$ and $H \circ b = H \circ c = H \circ d = H$.

The verification of the associativity property sometimes can be very demanding, because in general, it requires n^3 checks, where $|H| = n$.
Let us show the identity $(b \circ c) \circ d = b \circ (c \circ d)$. Since $b \circ c = \{a, b, c\}$, then the hyperproduct $(b \circ c) \circ d$ is equal to $\{a, b, c\} \circ d$, which is further equal to $a \circ d \cup b \circ d \cup c \circ d = H$. Similarly, $b \circ (c \circ d) = b \circ \{c, d\} = b \circ c \cup b \circ d = H$. All other checkings of the associativity identities can be done in a similar way. Since the hypergroupoid $(H, \circ)$ satisfies both, the associativity and the reproducibility, then it is a hypergroup.

Remark 1.1. Notice that a hypergroup H such that $|x \circ y| = 1$, for any $x, y \in H$ is a group, while every group is a hypergroup.

Some well known examples of hypergroups are listed below.

Example 1.2. [32] Let $(G, \cdot)$ be a group, H be a normal subgroup of G, and for all $x, y \in G$, the hyperoperation is given with $x \circ y = xyH$. Then $(G, \circ)$ is a hypergroup.

Example 1.3. [32] Let the hyperoperation " $\circ$ " be defined on the set of real numbers as follows: $x \circ x = x$ for all $x \in \mathbb{R}$ and $x \circ y$ is the open interval between x and y. The hyperstructure $(H, \circ)$ is a hypergroup.

Definition 1.1. [13] *If H is a non-empty set and for all $x, y \in H$ it holds that $x \circ y = H$, then the hypergroup $(H, \circ)$ is called a total hypergroup.*

Analogously to subgroups and semigroups in classical algebra theory, in hypercompositional algebra we introduce subsemihypergroups and subhypergroups.

Definition 1.2. [13] *A non-empty subset K of a semihypergroup $(H, \circ)$ is called a subsemihypergroup if it is a semihypergroup. In other words, a non-empty set K of a semihypergroup $(H, \circ)$ is a subsemihypergroup if $K \circ K \subseteq K$. A non-empty subset L of a hypergroup $(H, \circ)$ is called a subhypergroup if it is a hypergroup.*

Definition 1.3. [13] *A subhypergroup K of a hypergroup $(H, \circ)$ is said to be conjugable if for all $x \in H$ there exists $y \in H$ such that $x \circ y \subseteq K$.*

The set H itself is a subhypergroup of H. We call all other subhypergroups as proper subhypergroups.

Example 1.4. [35] If $\mathbb{Z}$ is the set of integers and the hyperproduct on the set $\mathbb{Z} \times \mathbb{Z}$ is defined as $(a, b) \circ (c, d) = \{(a, b + d), (c, b + d)\}$, then the hyperstructure $(\mathbb{Z} \times \mathbb{Z}, \circ)$ is a hypergroup, while the hyperstructure $(\mathbb{Z} \times \{0\}, \circ)$ is a subhypergroup of $(H, \circ)$.

Definition 1.4. [13] *Let $(H, \circ)$ be a hypergroupoid. An element e is called a left identity if for any $a \in H, a \in e \circ a$. Similarly, an element e is called a right identity if for any $a \in H, a \in a \circ e$. An element e is called an identity (or unit) if it is both, a left and a right identity, i.e., if for any $a \in H, a \in a \circ e \cap e \circ a$.*

Definition 1.5. [13] *Let $(H, \circ)$ be a hypergroup endowed with at least an identity. An element $a' \in H$ is called a left inverse of a if there exists an identity $e \in H$ such that $e \in a' \circ a$. An element $a' \in H$ is called a right inverse of a if there exists an identity $e \in H$ such that $e \in a \circ a'$. An element $a' \in H$ is called an inverse of a if it is both, a left and a right inverse, i.e., there exists an identity e such that $e \in a \circ a' \cap a' \circ a$.*

Definition 1.6. [13] *A hypergroup $(H, \circ)$ is called a regular hypergroup if it has at least one identity and all elements from H have at least one inverse.*

The first construction of a homomorphism was given by Corsini in [6], 34 years after the notion of a hypergroup was introduced. Later, in 1991, Jantosciak gave the definitions for the various types of homomorphisms [41]. We will present some of important definitions of homomorphism which are widely used in the study of hyperstructures.

Definition 1.7. [32] *Let $(H_1, \circ)$ and $(H_2, \star)$ be two hypergroupoids. A mapping $f : H_1 \to H_2$ is called*

1. *A homomorphism if for all $x, y \in H$ $f(x \circ y) \subseteq f(x) \star f(y)$.*
2. *A good homomorphism if for all $x, y \in H$ $f(x \circ y) = f(x) \star f(y)$.*
3. *A very good homomorphism if it is good and for all $x, y \in H$ we have $f(x/y) = f(x)/f(y)$ and $f(x \setminus y) = f(x) \setminus f(y)$ where $x/y = \{z \in H : x \in z \circ y\}$ and $x \setminus y = \{u \in H : y \in x \circ u\}$.*

There are many classes of hypergroups in hypergroup theory. We will mention some of them, which are relevant for our research. One of the most important classes of hypergroups are join spaces. Join spaces are introduced in [56] by Prenowitz. They are particular type of hypergroups, used in Graph theory, Geometry, Binary relations and other areas. Jantosciak and Prenowitz [57, 58] have given an algebraic interpretation of linear, spherical and projective geometry using "join" hyperoperation. In the linear geometry, the "join" hyperoperation assigns to two distinct points a segment, in the projective geometry it assigns to them a line, while in the spherical geometry, the "join" hyperoperation assigns to two distinct points a minor arc of great circle throught these points. Besides, join spaces can be used to characterize lattices, median algebras, graphs and so on.
Let a, b are elements from $(H, \circ)$, and denote

$$a/b = \{x \in H : a \in x \circ b\}.$$

The set a/b is called the quotient of a and b or the extension of a from b [32].

Definition 1.8. [32] *A commutative hypergroup $(H, \circ)$ is called a join space if for all a, b, c, d from H, there is*

$$a/b \cap c/d \neq \emptyset \to a \circ d \cap b \circ c \neq \emptyset$$

The particular type of join hypergroup having a scalar identity is called a canonical hypergroup. It was introduced by Krasner, who introduced them as an additive part of hyperrings and hyperfields. However, they were named after Mittas in [54], who has been later studied them in depth.

Definition 1.9. [32] *We say that $(H, \circ)$ is a canonical hypergroup if*

1. *It is a commutative*
2. *It has a scalar identity (scalar unit) i.e., there exists $e \in H$ such that for all $x \in H$ there is $x \circ e = e \circ x = x$.*
3. *Every element has a unique inverse, i.e., for all $x \in H$, there exists a unique $x^{-1} \in H$, such that $e \in x \circ x^{-1} \cap x^{-1} \circ x$.*
4. *It is reversible, which means that for any $(x, y, a) \in H^2$ holds*
   - (a) *if $y \in a \circ x$, then there exists $a^{'}$ inverse of a such that $x \in a^{'} \circ y$.*
   - (b) *if $y \in x \circ a$, then there exists $a^{''}$ inverse of a such that $x \in y \circ a^{''}$.*

Remark 1.2. [32] The identity of a canonical hypergroup is unique.

Let $(H, +)$ be a canonical hypergroup and N be an arbitrary canonical subhypergroup of H and set $H/N = \{x + N, x \in H\}$. Let us define the hyperoperation $+^{'}$ on H/N as follows

$$(x + N) +^{'} (y + N) = \{t + N | t \in x + y\}.$$

Proposition 1.1. [61] *For every canonical hypergroup H, if N is an arbitrary canonical subhypergroup of it, then the hypergroup $(H/N, +^{'})$ is a canonical, too.*

1.1.1 Corsini hypergroups

Let us present now a new class of hypergroups, called Corsini hypergroups. We will observe in depth the reducibility property for Corsini hypergroups in the third chapter. In the first studies concerning the relationship between hypergroups and hypergraphs, Corsini defined the following hypergroupoid.

Definition 1.10. [29] *Let $\Gamma = (H; \{A_i\}_i)$ be a hypergraph, i.e., for any i, $A_i \in \mathcal{P}(H) \setminus \emptyset; \bigcup_i A_i = H$ for any $x \in H$. Set $E(x) = \bigcup_{x \in A_i} A_i$. The hypergroupoid $H_\Gamma = (H, \circ)$ where the hyperoperation $\circ$ is defined by:*

$$\forall (x, y) \in H^2, \quad x \circ y = E(x) \cup E(y)$$

is called a hypergraph hypergroupoid.

Definition 1.11. [15] *The hypergroupoid H_Γ satisfies for each $(x, y) \in H^2$, the following conditions:*

1. $x \circ y = x \circ x \cup y \circ y$,

2. $x \in x \circ x$,

3. $y \in x \circ x$ *if and only if* $x \in y \circ y$.

Theorem 1.1. [15] *A hypergroupoid* $(H, \circ)$ *satisfying the conditions in Definition 1.11 is a hypergroup if and only if also the following condition is valid:*

$$\forall (a, c) \in H^2 \quad c \circ c \circ c \setminus c \circ c \subseteq a \circ a \circ a.$$

This hypergroup was studied also in [2], where the authors named it "Corsini hypergroup" and investigated also its properties connected with the Cartesian product. Here we recall one result, that we will need in our research.

Theorem 1.2. [2] *Let* $(H, \circ_1)$ *and* $(H, \circ_2)$ *be two Corsini hypergroups. Then the direct product of hypergroups* $(H \times H, \circ_1 \times \circ_2)$ *is a Corsini hypergroup if and only if* $(H, \circ_1)$ *or* $(H, \circ_2)$ *(or both) is a total hypergroup.*

Note that, for two given hypergroups defined on the same support set H, the hyperoperation $\otimes = \circ_1 \times \circ_2$ is defined as $(x_1, x_2) \otimes (y_1, y_2) = (x_1 \circ_1 y_1, x_2 \circ_2 y_2)$, $x_1, x_2, y_1, y_2 \in H$. The structure $(H \times H, \otimes)$ is called the *direct product of hypergroups*.

Let us define a particular type of Corsini hypergroup, studied for its important properties in the theory of automata and languages [52], which is called *B-hypergroup* by G. Massouros. The name of this hypergroup was given due to the binary result that the hyperoperation gives. It was also investigated in connection with fortified join spaces [51] or breakable semihypergroups [40].

Definition 1.12. [52] *Let* H *be any non-empty set. For any* $(x, y) \in H^2$, *define* $\star$ *as follows*

$$x \star y = \{x, y\}.$$

Then the hypergroup $(H, \star)$ *is called a B-hypergroup.*

Proposition 1.2. [2] *Any B-hypergroup* $(H, \star)$ *is a Corsini hypergroup.*

1.1.2 Fundamental relations in hypergroups

In the following we introduce one of the key concepts in the hypercompositional algebra. We define fundamental relations, which play the role of connection between the classical and the hypercompositional algebra.

As we have already explained in the introductory part, the algebraic hypergroups are the most natural generalization of the classical groups: the binary operation of groups is extended to a hyperoperation, where the composition of two elements of a given set gives a non-empty subset of it. The first example of such hyperoperation was given by Marty [49], when he noticed that if G is a group, and H is its subgroup, then the quotient G/H is a hypergroup. The quotient G/H forms a group only in the case when H is a normal subgroup. In classical algebra, quotient sets are important because they provide a tool for obtaining a stricter structure from the initial one. In the hyperalgebra, quotients sets are very important because they connect classical algebraic structures with algebraic hyperstructures. Connection between semihypergroups (hypergroups) and semigroups (groups) can be established via specific equivalence relations. These relations play a role analogous to the congruences in the classical algebra. If we start with a (semi) hypergroup, using this equivalence relation and a corresponding operation we get a (semi) group structure on the quotient set. To be more precize, equivalence relation defined on a hyperstructure such that the quotient set (hyperstructure modulo this equivalence relation) is a classical structure having the same behaviour, is called a *fundamental relation.* Besides, the fundamental relation is the smallest equivalence relation such that the described quotient set is a classical structure. The corresponding quotient sets are called *fundamental structures.* Using fundamental relations, algebraic hyperstructures can use a plenty of tools used in a classical algebra.
In order to give a strict definition for these relation, let us first define a strongly regular relation.

Definition 1.13. [13] *Let $(H,\circ)$ be a hypergroupoid, $a,b\in H$ and ρ be an equivalence relation on H. Then ρ is regular to the left if:*

$$\begin{array}{c} a\rho b \Rightarrow (\forall u\in H, \forall x\in u\circ a, \exists y\in u\circ b: \quad x\rho y \\ and \\ \forall u\in H, \forall y^{'}\in u\circ b, \exists x^{'}\in u\circ a: \quad x^{'}\rho y^{'}) \end{array} \tag{1.2}$$

The relation ρ is regular to the right if:

$$\begin{array}{c} a\rho b \Rightarrow (\forall u\in H, \forall x\in a\circ u, \exists y\in b\circ u: \quad x\rho y \\ and \\ \forall u\in H, \forall y^{'}\in b\circ u, \exists x^{'}\in a\circ u: \quad x^{'}\rho y^{'}) \end{array} \tag{1.3}$$

The relation ρ is regular if it is regular to the left and to the right.

Definition 1.14. [13] *Let $(H, \circ)$ be a hypergroupoid, $a, b \in H$ and ρ be an equivalence relation on H. Then ρ is strongly regular to the left if:*

$$a\rho b \Rightarrow \forall u \in H, \forall x \in u \circ a, \forall y \in u \circ b : \quad x\rho y.$$

The relation ρ is strongly regular to the right if:

$$a\rho b \Rightarrow \forall u \in H, \forall x \in a \circ u, \forall y \in b \circ u : \quad x\rho y.$$

The relation ρ is strongly regular if it is strongly regular to the left and to the right.

Given a semihypergroup H and a regular relation R, the quotient H/R is a semihypergroup. Besides, with a properly defined hyperoperation on the structure H/R, if the relation R is a strongly regular relation, then the quotient H/R is a semigroup.

Theorem 1.3. [32] *Let $(H, \circ)$ be a semihypergroup and R be an equivalence relation on H.*

1. *If R is regular, then the quotient H/R is a semihypergroup with respect to the following hyperoperation $\bar{x} \otimes \bar{y} = \{\bar{z} : z \in x \circ y\}$.*
2. *If the above hyperoperation is well defined on H/R, then the relation R is regular.*

Corollary 1.1. [13] *If $(H, \circ)$ is a hypergroup and R is an equivalence relation on H, then R is regular if and only if $(H/R, \otimes)$ is a hypergroup.*

The following theorem states that a semihypergroup H factorized by a strongly regular relation R is a semigroup.

Theorem 1.4. [13] *Let $(H, \circ)$ be a semihypergroup and R be an equivalence relation on H.*

1. *If R is strongly regular, then the quotient H/R is a semigroup with respect to the following operation $\bar{x} \otimes \bar{y} = \{\bar{z} : z \in x \circ y\}$.*
2. *If the above operation is well defined on H/R, then the relation R is strongly regular.*

Corollary 1.2. [32] *If $(H, \circ)$ is a hypergroup and R is an equivalence relation on H, then R is strongly regular if and only if $(H/R, \otimes)$ is a group.*

Strictly speaking, the fundamental relation is the smallest strongly regular equivalence relation, such that the corresponding hyperstructure factorized by this relation becomes a classical structure. Until now, for semi (hypergroups), two fundamental relations are defined, by Koskas [46] and Freni [39]. Later, this concept has been studied by Corsini, Vougiouklis, Davvaz, Loreanu-Fotea, Migliorato and many others. In 1970, Koskas connected classical structures with hyperstructures using a relation β. He noticed a similar behaviour of elements belonging to the same hyperproducts and using that, he defined a relation β which was reflexive and symmetric. After that, he denoted by β^* its transitive closure in order to define equivalence relation and to partition the quotient set into equivalence classes.

In the following we give the definition for the β relation.

Definition 1.15. [46] *Let $(H, \circ)$ be a semihypergroup and $n \geq 1, n \in \mathbb{N}$. We define the β_n relation as follows*

$$x\beta_n y \quad \textit{if there exist} \quad a_1, a_2, \ldots, a_n \quad \textit{such that} \quad \{x, y\} \subseteq \prod_{i=1}^{n} a_i$$

and let

$$\beta = \cup_{n\geq 1}\beta_n, \quad \textit{where} \quad \beta_1 = \{(x, x) | x \in H\}$$

is the diagonal relation on H.

The relation β is reflexive and symmetric [46]. We will denote with β^* the transitive closure of β.

Theorem 1.5. [46] *β^* is the smallest strongly regular equivalence relation on H with respect to the inclusion.*

Theorem 1.6. [46] *Let $(H, \circ)$ be a semihypergroup (hypergroup), then the relation β^* is the smallest equivalence relation such that the quotient H/β^* is a semigroup (group).*

As we have already mentioned, the relation β^* is called the *fundamental relation* on H and the quotient H/β^* is called the *fundamental semigroup (group)*. It is important to emphasize that in hypergroups, the fundamental relation β coincides with the β^* relation [32]. Thus, the quotient set obtained by factorizing a hypergroup by the equivalence β is a group.

Another fundamental relation, denoted γ, was defined on a semihypergroup by Freni. He denoted by γ^* its transitive closure, and he set $\gamma = \bigcup_{n\geq 1} \gamma_n$, where γ_1 is the diagonal relation and for, $n \geq 1, \gamma_n$ is the relation defined as follows [39]:

$$x\gamma_n y \Leftrightarrow \exists(z_1, z_2, \cdots, z_n) \in H^n : \exists\delta \in S_n : x \in \prod_{i=1}^{n} z_i, y \in \prod_{i=1}^{n} z_{\delta(i)}.$$

γ is symmetric and reflexive.

Theorem 1.7. [39] *Let H be a semihypergroup. The relation γ^* is the smallest strongly regular equivalence relation such that the quotient H/γ^* is a commutative semigroup.*

1.1.3 Complete hypergroups

Using fundamental relations, we define wide and very important class of hypergroups, called complete hypergroups.

The definition of a complete hypergroup is based on the notion of *complete part*, introduced by Koskas in [46]. The complete part is used for the purpose of characterization of the equivalence class of an element under the relation β^*. More precizely, a non-empty set A of a semihypergroup $(H, \circ)$ is called a *complete part* of H, if for any natural number n and any elements $a_1, a_2, \ldots, a_n$ in H, the following implication holds [46]:

$$A \cap \prod_{i=1}^{n} a_i \neq \emptyset \Rightarrow \prod_{i=1}^{n} a_i \subseteq A.$$

We may say, as it was mentioned in the overview paper written by Antampoufis et al [3], that a complete part A absorbs all hyperproducts of the elements of H having non-empty intersection with A. The intersection of all complete parts of H containing the subset A is called the *complete closure* of A in H and denoted by $C(A)$ [46].

The complete parts were later studied by Corsini [8] and Sureau [65]. De Salvo studied some of their properties in [37]. Migliorato also introduced a notion of a $n-$complete part, which is the generalization of complete parts [53].

For a given semihypergroup H and a strongly regular relation R on H, the equivalence class of any element x from H is a complete part of H [32].

Theorem 1.8. [32] *Let $(H, \circ)$ be a semihypergroup. The following conditions are equivalent:*

1. $\forall x, y \in H \quad \forall a \in x \circ y \quad C(a) = x \circ y.$
2. $\forall x, y \in H \quad C(x \circ y) = x \circ y.$

Definition 1.16. [32] *A semihypergroup is complete if it satisfies one of the above equivalent conditions. A hypergroup is complete if it is a complete semihypergroup.*

Let us define the notion of the heart of hypergroup, which is directly connected with fundamental relations.

Definition 1.17. [32] *Let* $(H, \circ)$ *be a hypergroup and consider the canonical projection* $\varphi_H : H \to H/\beta^*$. *The heart of the hypergroup* H *is the set* $\omega_H = \{x \in H | \varphi_H(x) = 1\}$, *where* 1 *is the identity of the group* $(H/\beta^*, \otimes)$.

As we explained before, β^* is the smallest strongly regular equivalence relation such that the quotient H/β^* represents a group and the operation $\otimes$ is given with

$$\beta^*(x) \otimes \beta^*(y) = \beta^*(z), z \in x \circ y, \quad \text{with} \quad x, y \in H.$$

From the above definition, it is clear that the heart contains all elements x for which the equivalence class $\beta^*(x)$ is the identity in H/β^*.

The heart of a hypergroup was studied in depth by Loreanu in her Phd thesis, and together with Corsini in [18].

Theorem 1.9. [32] *The heart* w_H *is a complete part of* H.

Moreover, the heart of a hypergroup is the smallest complete part of hypergroup H, which is also a subhypergroup of H[32].

Since it is satisfied that $\beta^*(x) = \omega_H \circ x = x \circ \omega_H$, we may say that the heart gives us an information about the partition set corresponding to the element x under the relation β^*. The heart ω_H of a complete hypergroup $(H, \circ)$ has an interesting property: it contains all identities of H.

Theorem 1.10. [13] *Let* $(H, \circ)$ *be a complete hypergroup.*

1. *The heart* ω_H *is the set of two-sided identities of* H.
2. H *is regular and reversible.*

As we can see in [13, 19, 30], in practice, it is more convinient to use the following characterization of the complete hypergroups.

Theorem 1.11. [13] *Any complete hypergroup may be constructed as the union* $H = \bigcup_{g \in G} A_g$ *of its subsets, where*

1) $(G, \cdot)$ *is a group.*

2) *The family* $\{A_g, | g \in G\}$ *is a partition of* G, *i.e., for any* $(g_1, g_2) \in G^2$, $g_1 \neq g_2$, *there is* $A_{g_1} \cap A_{g_2} = \emptyset$.

3) *If* $(a, b) \in A_{g_1} \times A_{g_2}$, *then* $a \circ b = A_{g_1 g_2}$.

Above theorem clearly shows that any group is a complete hypergroup, too. However, in the thesis, we will consider only proper complete hypergroups, so complete hypergroups that are not groups.

Example 1.5. [19] *Let* $(H, \circ)$ *be the hypergroup represented by the following commutative Cayley table:*

$\circ$	e	a_1	a_2	a_3
e	e	a_1, a_2, a_3	a_1, a_2, a_3	a_1, a_2, a_3
a_1		e	e	e
a_2			e	e
a_3				e

(1.4)

The hypergroup $(H, \circ)$ *is complete, where the group* $G = (\mathbb{Z}_2, +)$, *and the partition set contains* $A_0 = \{e\}$, $A_1 = \{a_1, a_2, a_3\}$. *It is easy to see that* H *is the union of sets* A_0 *and* A_1, *which are disjoint. Obviously,* $e \circ e = A_{0+0} = A_0 = e$. *Further,* $e \circ a_i = A_{0+1} = A_1$, *since* $e \in A_0, a_i \in A_1$, *for* $i \in \{1, 2, 3\}$. *Due to the commutativity,* $a_i \circ e = A_1$. *Further, because* $a_i \in A_1$, *for indices* $i \in \{1, 2, 3\}$, *then* $a_i \circ a_j = a_j \circ a_i = A_{1+1} = A_0 = e$. *Hence, all conditions in Theorem 1.11 are fulfilled.*

The complete hypergroups have been studied for their general properties [37], or in connection with their fuzzy grade [19, 26], or for their commutativity degree [62].

1.2 Hyperrings

Hyperrings are hypercompositional structures endowed with two (hyper)operations, and denoted additively and the other one multiplicatively (but not both operations), with similar properties of the operations on rings. There are different types of hyperring structures depending on how the addition and multiplication are defined, i.e., if they are defined as operations or hyperoperations. There are different concept of the hyperring structures in the hyperstructure theory. The hyperrings can be defined with the help of two hyperoperations, or with the one hyperoperation and the one operation. We differ between three types of hyperrings: additive, multiplicative and general hyperring. The additive hyperring is a hyperstructure endowed with an additive hyperoperation and multiplicative operation, where the multiplicative operation is distributive with respect to the additive hyperoperation. The most known additive hyperring was introduced by

Krasner in 1983 [47], and it was named after the author. Later, Krasner also studied quotient hyperrings and hyperfields. This type of hyperring has been widely studied by many authors, as Massouros, Loreanu-Fotea, Davvaz, Mittas, Vougiouklis, Spartalis and others. The other two types of hyperrings are multiplicative and general hyperring.

Definition 1.18. [47] *A Krasner hyperring is an algebraic structure* $(R, +, \cdot)$ *such that the additive part* $(R, +)$ *is a canonical hypergroup, the multiplicative part* $(R, \cdot)$ *is a semigroup having zero as a bilaterally absorbing element, i.e.,* $x \cdot 0 = 0 \cdot x = 0$ *for all* $x \in H$, *and the multiplication "$\cdot$" is distributive with respect to the hyperoperation "$+$".*

A Krasner hyperring is commutative if $(R, \cdot)$ is a commutative semigroup. A Krasner hyperring is a hyperring with unit if the semigroup $(R, \cdot)$ has a unit [47].

Example 1.6. [4] On the set $R = \{0, a, b, c\}$, define an hyperoperation $+$ and a multiplication $\cdot$ by the following tables:

$+$	0	a	b	c
0	0	a	b	c
a	a	$0, b$	a, c	b
b	b	a, c	$0, b$	a
c	c	b	a	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	b	b	0
c	0	c	0	c

The structure $(R, +, \cdot)$ is a Krasner hyperring.

Example 1.7. [32] If $(H, \leq, +)$ is a totally ordered group and the hyperaddition is given with

$$x \oplus x = \{t \in H | t \leq x\}, \forall x \in H,$$

$$x \oplus y = \{max\{x, y\}\}, \forall x, y \in H, x \neq y,$$

then the structure $(H, \oplus)$ defines a canonical hypergroup. If $(H, +, \cdot)$ is a totally ordered ring, then $(H, \oplus, \cdot)$ is a Krasner hyperring.

Definition 1.19. [34] *A commutative Krasner hyperring with unit is called a Krasner hyperfield if* $R \setminus \{0\}$ *is a group.*

Example 1.8. [5] On the set $F = \{0, 1\}$ define an additive hyperoperation "$+$" and a multiplicative operation "$\cdot$" by the following tables:

$+$	0	1
0	0	1
1	1	$0, 1$

$\cdot$	0	1
0	0	0
1	0	1

The hyperstructure $(F, +, \cdot)$ is a Krasner hyperfield.

An important example of a Krasner hyperfield can be found in [47], where the author presented the way to construct Krasner hyperfields using a field.

Example 1.9. [47] Let $(F, +, \cdot)$ be a field, G be a subgroup of $(F \setminus \{0\}, \cdot)$ and let $H = F/G = \{aG | a \in F\}$ where the hyperaddition and the multiplication are given with the formulas:

$$aG \oplus bG = \{cG | c \in aG + bG\},$$

$$aG \odot bG = abG.$$

Then the hyperstructure $(H, \oplus, \odot)$ is a hyperfield.

Definition 1.20. [32] *A subhyperring of a Krasner hyperring $(R, +, \cdot)$ is a non-empty subset A of R which forms a Krasner hyperring.*

Definition 1.21. [32] *Let $(R, +, \cdot)$ be a hyperring, and A be a subhyperring of R. We say that A is a left (right) hyperideal of R if $r \cdot a \in A \quad (a \cdot r \in A)$ for all $r \in R$, with $a \in A$. A is a hyperideal of R if it is both, left and right hyperideal.*

In practice, sometimes it is more suitable to use the following characterization.

Lemma 1.1. [32] *Let A be a non-empty set of the hyperring $(R, +, \cdot)$. A is a left (right) hyperideal of the hyperring if and only if*

1. *For any $a, b \in A$, it holds that $a - b \in A$.*

2. *If $a \in A, r \in R$ then $r \cdot a \in A \quad (a \cdot r \in A)$.*

Example 1.10. [5] Let $(R, +, \cdot)$ be the hyperring from Example 1.6. The hyperideals of the hyperring R are the sets: $\{0\}, \{0, b\}, \{0, c\}, \{0, b, c\}$ and R.

The another type of hyperring, equipped with an additive operation and a multiplicative hyperoperation was introduced by R. Rota in [59]. This type of hyperring is called a multiplicative hyperring.
More exactly, the structure $(R, +, \cdot)$ is a *multiplicative hyperring* if: $(R, +)$ is an Abelian group, $(R, \cdot)$ is a semihypergroup, and the operation $\cdot$ is weakly distributive with respect to the hyperoperation $+$, i.e., $a(b + c) \subseteq ab + ac$ and $(b + c)a \subseteq ba + ca$ for all $a, b, c \in R$ [5].

Example 1.11. [32] Let K be a field and V be a vector space over K. Let $< a, b >$ be a subspace generated by the set $\{a, b\}$, where $a, b \in V$. Then, if we define for all

$$a, b \in V, \quad a \circ b =< a, b > .$$

Then the hyperstructure $(V, +, \circ)$ is a multiplicative hyperring.

Example 1.12. [32] Let $(R, +, \cdot)$ be a ring and I be an ideal of it. If we define the hyperoperation on R as

$$\forall a, b \in R \quad a \star b = ab + I,$$

then the hyperstructure $(R, +, \star)$ is a multiplicative hyperring.

Definition 1.22. [32] *Let $(R, +, \cdot)$ be a multiplicative hyperring and H be a non-empty subset of R. We say that H is a subhyperring of $(R, +, \cdot)$ if $(H, +, \cdot)$ is a multiplicative hyperring itself.*

Similarly to the Krasner hyperring, a non-empty subset A of a multiplicative hyperring R is a *left (right) hyperideal* if for all $a, b \in A$ there is $a - b \in A$ and if $a \in A, r \in R$ it implies that $r \cdot a \in A(a \cdot r \in A)$. If the hyperideal is both, left and right, it is called a *hyperideal* [32].

Example 1.13. [27] Let $(\mathbb{Z}_A, +, \circ)$ be a multiplicative hyperring where $\mathbb{Z}_A = \mathbb{Z}$, and the hypermultiplication is given with $x \circ y = \{x \cdot a \cdot y | a \in A\}$, where $A = \{2, 4\}$. Then the set $12\mathbb{Z} = \{12n : n \in \mathbb{Z}\}$ is a hyperideal of the hyperring $(\mathbb{Z}_A, +, \circ)$.

The widest class of hyperrings is the class of general hyperrings. These are hyperstructures endowed with two hyperoperations, connected by the distributibivity property. The general hyperring was firstly introduced by Corsini [7], who used it for defining and studied feeble hypermodules. Many authors gave a definition of a general hyperring, but the most general one was by Spartalis in 1989 [63]. In order to define general hyperrings let us first define the hyperringoid.

Definition 1.23. [52] *A hypercompositional structure $(R, \oplus, \odot)$ is called a hyperringoid if*

1. *$(R, \oplus)$ is a hypergroup.*
2. *$(R, \odot)$ is a semigroup.*
3. *The operation "$\odot$" distributes on both sides over the hyperoperation "$\oplus$."*

This algebraic hyperstructure was first introduced by Massouros and Mittas [52] in the study on languages and automata. If we request that both addition and multiplication are hyperoperations, then the hyperringoid becomes a general hyperring.

Definition 1.24. [67] *A triple $(R, \oplus, \odot)$ is a general hyperring if:*

1. *$(R, \oplus)$ is a hypergroup.*
2. *$(R, \odot)$ is a semihypergroup.*
3. *The multiplication $\odot$ is distributive with respect to the addition $\oplus$, i.e., for all $a, b, c \in R$, $\quad a \odot (b \oplus c) = a \odot b \oplus a \odot c$ and $(a \oplus b) \odot c = a \odot c \oplus b \odot c$.*

Definition 1.25. *[32] A commutative general hyperring $(R, \oplus, \odot)$ is called a hyperfield if $R^* \neq \emptyset$, where $R^* = R \setminus \{\omega\}, \omega$ is the heart of the additive part of a hyperring and $(R^*, \odot)$ is a hypergroup.*

Example 1.14. [34] The hyperstructure $(R, \oplus, \odot)$, where $R = \{a, b, c, d\}$ is a hyperfield.

$\oplus$	a	b	c	d
a	a	a, b	c, d	c, d
b	a, b	a, b	c, d	c, d
c	c, d	c, d	a, b	a, b
d	c, d	c, d	a, b	a, b

$\odot$	a	b	c	d
a	a, b	a, b	a, b	a, b
b	a, b	a, b	a, b	a, b
c	a, b	a, b	c, d	c, d
d	a, b	a, b	c, d	c, d

Definition 1.26. [32] *Let $(R, \oplus, \odot)$ be a general hyperring and let K be a non-empty subset of it. We say that K is a subhyperring of R if it satisfies the following conditions:*

1. *$(K, \oplus)$ is a subhypergroup of $(R, \oplus)$.*
2. *$(K, \odot)$ is a subsemihypergroup of $(R, \odot)$.*

Definition 1.27. *[32] Let $(R, \oplus, \odot)$ be a general hyperring and let I be a non-empty subset of it. We say that I is a left (right) hyperideal of R, if it satisfies the following conditions:*

1. *$(I, \oplus)$ is a subhypergroup of $(R, \oplus)$.*
2. *For all $x \in I, a \in R, a \odot x \subseteq I (x \odot a \subseteq I)$.*

I is a hyperideal if it is a left and right hyperideal.

Every hyperring has two trivial hyperideals, the heart of an additive part of a hyperring ω and a hyperring R.

Example 1.15. Notice that the subset $\{a, b\}$ is a hyperideal of the hyperfield presented in Example 1.14.

Definition 1.28. [32] *Let R_1 and R_2 be two general hyperrings. A mapping ϕ from $(R_1, +, \cdot)$ to $(R_2, \oplus, \odot)$ is said to be a good (strong) homomorphism if for all $a, b \in R_1$*

1. $\phi(a + b) = \phi(a) \oplus \phi(b)$.
2. $\phi(a \cdot b) = \phi(a) \odot \phi(b)$.
3. $\phi(0) = 0$.

The H_v- structures were introduced by Vougiouklis at the 4th AHA Congress in 1990 [69], as hypercompositional structures with weak associative hyperoperations.

Definition 1.29. [32] *The hyperstructure $(H, \cdot)$ is an H_v-semigroup if $x \cdot (y \cdot z) \cap (x \cdot y) \cdot z \neq \emptyset$ for all $x, y, z \in H$. If also the reproduction axiom is valid, i.e., $a \cdot H = H \cdot a = H$, for all $a \in H$ then $(H, \cdot)$ is an H_v-group.*

Definition 1.30. [32] *A multi-valued system $(R, \oplus, \odot)$ is an H_v-ring if:*

1. *$(R, \oplus)$ is an H_v- group.*
2. *$(R, \odot)$ is an H_v- semigroup.*
3. *The multiplication $\odot$ weakly distributes with respect to the addition $\oplus$, i.e., for all $a, b, c \in R$, $(a \odot (b \oplus c)) \cap (a \odot b \oplus a \odot c) \neq \emptyset$ and $((a \oplus b) \odot c) \cap (a \odot c \oplus b \odot c) \neq \emptyset$.*

It is important to recall here that the quotient of a group with respect to any of its subgroups is a hypergroup, while the quotient of a group by any equivalence relation gives birth to an H_v-group [50]. A recently published overview of the theory of weak-hyperstructures is covered in [70].

In the following we will recall the construction of three types of hyperrings, that we will study in the fourth chapter. The first one leads to an H_v- ring obtained from a ring. This structure was principally studied by Spartalis and Vougiouklis [64], in connection with homomorphisms and numeration.

Let $(R, +, \cdot)$ be a ring and P_1 and P_2 be non-empty subsets of R. The hyperoperations:

$xP_1^*y = x+y+P_1$ and $xP_2^*y = x\cdot y\cdot P_2$ for all $x, y \in R$ are called the $P-$ *hyperoperations* [68]. Let $Z(R)$ be the center of the multiplicative group $(R, \cdot)$.

Theorem 1.12. [64] *Let $(R, +, \cdot)$ be a ring and P_1 and P_2 be non-empty subsets of R. If $0 \in P_1$ and $Z(R) \cap P_2 \neq \emptyset$, then (R, P_1^*, P_2^*) is an H_v- ring, called the H_v- ring with the $P-$ hyperoperations.*

We finish this section by recalling the construction of the hyperring of the formal series [31, 43]. Based on this, the structure of the set of polynomials over hyperring was studied.

Let $(R, +, \cdot)$ be a general commutative hyperring.

[31] A formal power series with coefficients in R is an infinite sequence $(a_0, a_1, a_2, \ldots, a_n, \ldots)$ of elements a_i in R. The set of all such power series is denoted by $R[[x]]$. We say that two power series $(a_0, a_1, a_2, \ldots, a_n, \ldots)$ and $(b_0, b_1, b_2, \ldots, b_n, \ldots)$ are equal if and only if $a_i = b_i$ for all indices i.

Let define on $R[[x]]$ the addition by

$$(a_0, a_1, a_2, \ldots, a_n, \ldots) \oplus (b_0, b_1, b_2, \ldots, b_n, \ldots) =$$

$$\{(c_0, c_1, c_2, \ldots, c_n, \ldots), c_k \in a_k + b_k\}$$

and the multiplication by

$$(a_0, a_1, a_2, \ldots, a_n, \ldots) \odot (b_0, b_1, b_2, \ldots, b_n, \cdots) =$$

$$\{(c_0, c_1, c_2, \ldots, c_n, \ldots), c_k \in \sum_{i+j=k} a_i \cdot b_j\}$$

The structure $(R[[x]], \oplus, \odot)$ is a general hyperring. It is worth to recall that the set of the polynomials $R[x]$ with coefficients in R is a left superring, with the same hyperoperations $\oplus$ and $\odot$ defined above. This means that $(R[x], \oplus)$ is a canonical hypergroup, $(R[x], \odot)$ is a semihypergroup with 0 a bilaterally absorbing element and the multiplication is weakly distributive on the left side with respect to the addition, i.e., $f \odot (g \oplus h) \subseteq f \odot g \oplus f \odot h$, for $f, g, h \in R[x]$.

1.2.1 Fundamental relations in hyperrings

Similar as for hypergroups, Vougiouklis intoduced the α^*- relation, which is the smallest equivalence relation such that the quotient set (the hyperring modulo the relation) is a ring, and he was the first one who named such a relation *fundamental*. He also investigated its relationship with the β^*- relation.

Definition 1.31. [32] *Let $(R,+,\cdot)$ be a general semihyperring (hyperring). We define α as follows:*

$$a\,\alpha\,b \quad \textit{iff} \quad \{a,b\}\subset u \quad \textit{where} \quad u \quad \textit{is a finite sum of finite products of elements from } R.$$

This is a reflexive and symmetric relation, but generally not transitive [32]. The transitive closure α^* of the relation α is called the *fundamental relation* on R.

Let us denote by U the set of all the finite sums of products of elements of R, and with $\alpha^*(a)$ the fundamental class of a. Then [32]

$$a\,\alpha\,b \quad \text{iff} \quad \exists z_1,\ldots,z_{n+1}\in R \qquad with \quad z_1=a, z_{n+1}=b \quad \text{and}$$

$$u_1,u_2,\ldots,u_n\in U \quad \text{such that} \quad \{z_i,z_{i+1}\}\subset u_i, \quad \text{for} \quad i=1,\ldots,n.$$

Theorem 1.13. [32] *Let $(R,+,\cdot)$ be a hyperring. Then the relation α^* is the smallest equivalence relation defined on R such that R/α^* is a ring. The quotient R/α^* is called the fundamental ring.*

1.3 Fuzzy sets and connections with hyperstructures

Fuzzy sets have been introduced by L.A. Zadeh in 1965 [71] and they represent the extension of the classical notion of a set. Any fuzzy set is characterized by a membership function which assigns to every element a degree of membership. In the classical set theory an element belongs or does not belong to the set, which means that the membership function is a binary function. More exactly, it is the characteristic function of the given set, so it maps every element to 0 or 1, depending if the element doesn't belong or belongs to the set. In the fuzzy set theory, the membership function maps every element to a number from the interval $[0,1]$.

Definition 1.32. [32] *Let X be a set. A fuzzy subset A of X is characterized by a membership function $\mu_A : X \to [0,1]$ which associates with each point $x \in X$ its grade or degree of membership $\mu_A(x) \in [0,1]$.*

Example 1.16. [38] On the set X of real numbers, consider one initial point a and one final point c, while b represents one intermediate point, i.e., $a < b < c$. The membership function $\mu_A(x)$ is defined as follows:

$$\mu_A(x) = \begin{cases} 0, & if\, x \leq a \\ \frac{x-a}{b-a}, & if\, a \leq x \leq b \\ \frac{c-x}{c-b}, & if\, b \leq x \leq c \\ 0, & if\, x \geq c \end{cases} \tag{1.5}$$

The membership function can be represented as in Figure 1.1. Because its triangular form, the fuzzy set $(X, \mu_A(x))$ is known as the triangular fuzzy number.

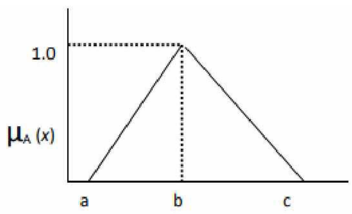

FIGURE 1.1: Triangular fuzzy number

Example 1.17. *On the set $X = \mathbb{N}$, let $A =$ "the set of natural numbers closed to 10". This fuzzy set can be represented by its membership function $\mu_A : X \to [0,1]$, where $\mu_A(x) = 1 - \frac{|10-x|}{10}$, which gives:*
$\mu_A(10) = 1, \mu_A(9) = \mu_A(11) = 0.9, \mu_A(7) = \mu_A(12) = 0.8$ and so on.

Definition 1.33. [29] *Let A, B be fuzzy subsets of X. Define the following operations:*
$A \subseteq B \Leftrightarrow \mu_A(x) \leq \mu_B(x), \forall x \in X.$
$A = B \Leftrightarrow \mu_A(x) = \mu_B(x), \forall x \in X.$
$C = A \cup B \Leftrightarrow \mu_C(x) = max\{\mu_A(x), \mu_B(x)\}, \forall x \in X.$
$C = A \cap B \Leftrightarrow \mu_C(x) = min\{\mu_A(x), \mu_B(x)\}, \forall x \in X.$
${\mu_A}^C(x) = 1 - \mu_A(x), \forall x \in X.$

The connections between hyperstructures and fuzzy sets can be approached in three ways. First, we can define crisp hyperoperations trought fuzzy sets, as it was done by Corsini in [14]. The second approach are fuzzy hyperalgebras, which can be considered as the extensions of the concept of fuzzy algebraic structures. For example, let the hypergroup $(H, \circ)$ be a crisp hypergroup, and μ be a fuzzy subset on it. We say that the fuzzy set μ is a fuzzy subhypergroup of $(H, \circ)$ if every level set of the fuzzy set μ is a subhypergroup of $(H, \circ)$ [29]. Recall that if μ is a fuzzy subset of a set H, then the *level set* of μ, noted with μ_t defines as: $\mu_t = \{x \in H | \mu(x) \geq t\}$, where t belongs to $[0, 1]$. In [28], where Davvaz introduced the concept of fuzzy subhypergroups, he also introduced the concept of the fuzzy H_v-subgroup of an H_v-group.

The third approach refers to fuzzy hypergroups, such that fuzzy hyperoperation assigns to any two elements a fuzzy set. They were studied by Corsini, Zahedi, Davvaz and many others. Here, the fuzzy hyperoperation associates to every pair of elements a fuzzy set, instead of the non-empty subset.

1.3.1 Construction of join spaces using fuzzy sets

Let us explain in more detail the first approach involving the very important connection between fuzzy sets and hyperstructures given by Corsini in [14]. With any fuzzy subset defined on a non-empty set H, he associates a join spaces.

Theorem 1.14. [14] *Let $\mu : H \to [0, 1]$ be a fuzzy subset of H, where H is a nonempty set. Defining*

$$x \circ y = \{z \in H : \mu(x) \wedge \mu(y) \leq \mu(z) \leq \mu(x) \vee \mu(y)\} \tag{1.6}$$

The hypergroup $(H, \circ)$ is a join space.

Conversely, Corsini defined a fuzzy subset associated with a hypergroupoid $(H, \circ)$ as follows [16]:

$$\widetilde{\mu}(u) = \frac{A(u)}{q(u)},$$

where $A(u) = \sum\limits_{(x,y) \in Q(u)} \frac{1}{|x \circ y|}, Q(u) = \{(x, y) \in H^2 : u \in x \circ y\}, q(u) = |Q(u)|$. For $Q(u) = \emptyset$, by default we take $\widetilde{\mu}(u) = 0$. We can interpret $\widetilde{\mu}$ as the average of the reciprocals of the sizes of the hyperproducts $x \circ y$ containing u [29]. By associating a fuzzy set $\widetilde{\mu}$ to the hypergroupoid as in formula 1.6, we obtain the join space $({}^1H, \circ_1)$. Using formula 1.6 once again, from $({}^1H, \circ_1)$ we get the associated join space $({}^2H, \circ_2)$ and obtain a new membership function $\widetilde{\mu}_2$. By this procedure we get a sequence of join spaces $(({}^iH, \circ_i), \widetilde{\mu}_i)_{i \geq 1}$ associated with H.

The length of the sequence of join spaces associated with H, i.e., the number of non-isomorphic join spaces in sequence is called the *fuzzy grade* of the hypergroupoid H, and the fuzzy set $\widetilde{\mu}$ is called the grade fuzzy set [29].

Definition 1.34. [13] *A hypergroupoid H has the fuzzy grade $m, m \in \mathbb{N} \setminus 0$, and write $f.g.(H) = m$ if, for any i, $0 \le i \le m$, the join spaces ${}^iH$ and ${}^{i+1}H$ associated with H are not isomorphic (where ${}^0H = H$) and for any $s, s \geq m,{}^s H$ is isomorphic with ${}^m H$.*

Let us show the above described procedure for a concrete hypergroup and calculate its fuzzy grade.

Example 1.18. Let the hypergroupoid $H = \{a, b, c\}$ be given with the following table:

$\circ$	a	b	c
a	a, b	b	b, c
b	a	a, b	b, c
c	a, c	b	b

(1.7)

The values of the elements a, b, c through the grade fuzzy set $\widetilde{\mu}$ are:

$$\widetilde{\mu}(a) = \frac{5}{8}, \widetilde{\mu}(b) = \frac{5}{7}, \widetilde{\mu}(c) = \frac{1}{2}.$$

Let us now construct a new join space.
The hyperproduct $a \circ b$ is equal to $\{z \in H : \widetilde{\mu}(a) \wedge \widetilde{\mu}(b) \le \widetilde{\mu}(z) \le \widetilde{\mu}(a) \vee \widetilde{\mu}(b)\}$, which is further equal to $\{z \in H : \frac{5}{8} \le \widetilde{\mu}(z) \le \frac{5}{7}\} = \{a, b\}$. Similarly, $b \circ c = \{z \in H : \widetilde{\mu}(b) \wedge \widetilde{\mu}(c) \le \widetilde{\mu}(z) \le \widetilde{\mu}(b) \vee \widetilde{\mu}(c)\} = \{z \in H : \frac{1}{2} \le \widetilde{\mu}(z) \le \frac{5}{7}\} = \{a, b, c\} = H$.
If we apply the same procedure for all other hyperproducts it gives to us the first join space represented by the table

$\circ_1$	a	b	c
a	a	a, b	a, c
b	a, b	b	H
c	a, c	H	c

(1.8)

According to the definion of a join space, if $a/b \cap c/d \neq \emptyset$ then $a \circ d \cap b \circ c \neq \emptyset$ for all elements $a, b, c, d \in H$. Indeed, a/b contains all elements $x \in H$ such that a belongs to $x \circ b$, giving $a/b = \{a, c\}$. On the other side, a/c contains elements $\{x \in H : a \in x \circ c\}$ which is equal to $\{a, b\}$, i.e. $a/b \cap b/c = \{a\}$. If we calculate $a \circ c \cap b \circ c$ we get $\{a, c\} \cap H = \{a, c\}$, with $\{a, c\} \neq \emptyset$ which proves that the above condition is satisfied. Analogously we show that the condition is satisfied for all other combination of elements, concluding that the obtained structure represents the join space.

Now, we calculate using formula 1.6

$$\widetilde{\mu}_1(a) = \frac{2}{3}, \widetilde{\mu}_1(b) = \frac{11}{21}, \widetilde{\mu}_1(c) = \frac{8}{15}$$

and the second associated join space is represented by the table:

$\circ_2$	a	b	c
a	a	H	a, c
b	H	b	b, c
c	a, c	b, c	c

(1.9)

From here, we get

$$\widetilde{\mu}_2(a) = \widetilde{\mu}_2(b) = \widetilde{\mu}_2(c) = \frac{8}{15}.$$

It is clear now that the associated join space has the table:

$\circ_3$	a	b	c
a	H	H	H
b	H	H	H
c	H	H	H

(1.10)

Easily, one notices that any associated join space ${}^sH, s \geq 4$ is the same as ${}^3H$, so the fuzzy grade of the given hypergroup is 3, because the number of non-isomorphic join spaces associated with H is 3.

Let us now define a fuzzy hyperstructure, i.e., the hyperstructure endowed with a fuzzy hyperoperation. In [60], Sen and Ameri gave the definition of a fuzzy semihypergroup.

Definition 1.35. [60] *Let S be a non-empty set. A fuzzy hyperoperation on S is a mapping $\circ : S \times S \to F(S)$, where $F(S)$ is the set of all fuzzy subsets of S. The structure $(S, \circ)$ is called a fuzzy hypergroupoid.*

Theorem 1.15. [60] *A fuzzy hypergroupoid $(S, \circ)$ is called a fuzzy hypersemigroup if for all $a, b, c \in S$, $(a \circ b) \circ c = a \circ (b \circ c)$ where for any fuzzy subset μ of S there is*

$$(a \circ \mu)(r) = \begin{cases} \vee_{t \in S}((a \circ t)(r) \wedge \mu(t)), & \text{if} \quad \mu \neq \emptyset \\ 0, \text{otherwise} \end{cases} \tag{1.11}$$

$$(\mu \circ a)(r) = \begin{cases} \vee_{t \in S}(\mu(t) \wedge (t \circ a)(r), & \textit{if} \quad \mu \neq \emptyset \\ 0, \textit{otherwise} \end{cases} \tag{1.12}$$

for all r in S.

Definition 1.36. [60] *A fuzzy hypersemigroup $(S, \circ)$ is called a fuzzy hypergroup if $x \circ S = S \circ x = \chi_S$, for all x in S, where χ_S is the characteristic function of the set $S, i.e.,$*

$$\chi_S(x) = \begin{cases} 1, \textit{if} \quad x \in S \\ 0, \textit{if} \quad x \notin S. \end{cases} \tag{1.13}$$

Example 1.19. [60] Let $S = \{a, b\}$ and define the fuzzy hyperoperation as: $(a \circ a)(a) = 0.1, (a \circ a)(b) = 0.2, (a \circ b)(a) = 0.2, (a \circ b)(b) = 0.2, (b \circ a)(a) = 0.3, (b \circ a)(b) = 0.2, (a \circ b)(b) = 0.2, (b \circ b)(a) = 0.7, (b \circ b)(b) = 0.8.$
Let us check whether the hyperproduct $(a \circ a) \circ a$ is equal to $a \circ (a \circ a)$.

Here, $((a \circ a) \circ a)(r) = \vee_{t \in S}((a \circ a)(t) \wedge (t \circ a)(r)) =$

$$((a \circ a)(a) \wedge (a \circ a)(r)) \vee ((a \circ a)(b) \wedge (b \circ a)(r)).$$

This gives that $((a \circ a) \circ a)(a) = 0.1 \vee 0.2 = 0.2$, while $((a \circ a) \circ a)(b) = 0.1 \vee 0.2 = 0.2$. Similarly, $(a \circ (a \circ a))(r) = \vee_{t \in S}((a \circ t)(r) \wedge (a \circ a)(t)) =$

$$((a \circ a)(r) \wedge (a \circ a)(a)) \vee ((a \circ b)(r) \wedge (a \circ a)(b)).$$

At the same way we conclude that $(a \circ (a \circ a))(a) = 0.2$ and $(a \circ (a \circ a))(b) = 0.2$, which finally proves that $(a \circ a) \circ a$ is equal to $a \circ (a \circ a)$. The all other identities can be proved in the similar way. The structure $(S, \circ)$ is a fuzzy hypersemigroup.

Chapter 2

Reducibility in hypergroups

This chapter deals with the reducibility property in hypergroups. We introduce the concept of the reducibility and examine the reducibility in certain types of hypergroups.

2.1 Reducibility in hypergroups

The concept of reducibility was introduced by James Jantosciak in 1990 at the Fourth International AHA Congress [42]. He noticed that it may happen that the hyperoperation does not distinguish between two elements, i.e., that two elements have the same role with respect to the hyperoperation. He defined three equivalence relations in order to claster elements with the same behaviour and called them fundamental.

The fundamental relations defined by Jantosciak [42] on an arbitrary hypergroup are operational equivalence, inseparability and essential indistinguishability.

Definition 2.1. [42] *Two elements x, y in a hypergroup $(H, \circ)$ are called:*

1. *operationally equivalent or by short o-equivalent, and write $x \sim_o y$, if $x \circ a = y \circ a$, and $a \circ x = a \circ y$, for any $a \in H$;*
2. *inseparable or by short i-equivalent, and write $x \sim_i y$, if, for all $a, b \in H$, $x \in a \circ b \Longleftrightarrow y \in a \circ b$;*
3. *essentially indistinguishable or by short e-equivalent, and write $x \sim_e y$, if they are operationally equivalent and inseparable.*

Remark 2.1. *Although they have the same name, the fundamental relations defined by Jantosciak must not be confused with the fundamental relations defined in the previous* *section, which are also called fundamental and connect classical algebraic structures with hyperstructures.*

Definition 2.2. [42] *A reduced hypergroup has the equivalence class of each element with respect to the essentially indistinguishable relation $\sim_e$ a singleton, i.e., for any $x \in H$, there is $\hat{x}_e = \{x\}$.*

As we can see from the previous definition, if the equivalence class of any element $x \in H$ contains no elements except x, the hypergroup is called reduced. Otherwise, we call it a non-reduced hypergoup. Regarding the definition of a reduced hypergroup, we have to take care that $\hat{x}_e = \{x\}$ does not mean that the equivalent classes with respect to both, the operational equivalence and the inseparability are singleton. Moreover, it can happen that neither these two equivalence classes is singleton. Let us suppose that, for example, $\hat{x}_o = \{x, y\}$ and $\hat{x}_i = \{y, z\}$. From here, it follows that $\hat{x}_e = \hat{x}_o \cap \hat{x}_i = \{x\}$.

However, if the hypergroup H is not reduced, so there exist two elements which belong to the same equivalence class, i.e., $\hat{x}_e = \hat{y}_e = \{x, y\}$, then it neccesarily implies that $\hat{x}_o = \hat{y}_o \supseteq \{x, y\}; \hat{x}_i = \hat{y}_i \supseteq \{x, y\}$.

In the same paper, Jantosciak defined a reduced form of a hypergroup, i.e., he found a manner how to construct a new reduced hypergroup from the given one.

Proposition 2.1. [42] *For any hypergroup $(H, \cdot)$, the quotient $(H/\sim_e, \star)$ is a reduced hypergroup and it is called a reduced form of the hypergroup H.*

The quotient hypergroup $H/\sim_e$ contains equivalence classes $\hat{x}_e$ with $x \in H$ where $\hat{x}_e \star \hat{y}_e = \{\hat{z}_e : z \in xy\}$.

Proposition 2.2. [42] *Let f be a mapping from H onto a reduced hypergroup K, such that $x \cdot y = f^{-1}(f(x)f(y))$, for all $x, y \in H$. Then $K \cong H/\sim_e$.*

The above proposition characterizes a reduced form $H/\sim_e$ as the reduced hypergroup from which is possible to reconstruct the hypergroup H. According to this, as Jantosciak explained in [42], we can split the study of hypergroups into two parts, the study of reduced hypergroups and the study of hypergroups with the same reduced form.

The following proposition shows that a hyperoperation on H may be reconstructed from the hyperoperation on $H/\sim_e$ via the canonical mapping $f : H \to H/\sim_e$ where $x \cdot y = f^{-1}(f(x)f(y))$. Also, the proposition enables us to determine all hypergroups having the hypergroup H as their reduced form [42].

Proposition 2.3. [42] *Let H be a hypergroup and K be a set. Let f be a mapping from K onto H, such that $x \cdot y = f^{-1}(f(x)f(y))$, for all $x, y \in H$. Then K is a hypergroup and the hypergroup H is reduced if and only if $K/{\sim_e} \cong H$.*

The following example explains the role of these fundamental relations.

Example 2.1. [42] Define on the set $H = \mathbb{Z} \times \mathbb{Z}^*$, where $\mathbb{Z}$ is the set of integers and $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$, the equivalence $\sim$ that assigns equivalent fractions in the same class: $(x, y) \sim (u, v)$ if and only if $xv = yu$, for $(x, y), (u, v) \in H$. Endow H with a hypercompositional structure, considering the hyperproduct $(w, x) \circ (y, z) = \widehat{(wz + xy, xz)}_{\sim}$. It can be proved that the equivalence class of the element $(x, y) \in H$ with respect to all three fundamental relations is equal to the equivalence class of (x, y) with respect to the equivalence $\sim$. The equivalence class of an ordered pair (x, y) contains all order pairs (u, v), such that the fractions $\frac{u}{v}$ are equal to $\frac{x}{y}$. Therefore, H is not a reduced hypergroup, but its reduced form is isomorphic with $\mathbb{Q}$, the set of rationals [22].

In the following we give an example of a non-reduced hypergroup and its reduced form.

Example 2.2. Let $(H, \circ)$ be a hypergroup, where the hyperoperation " $\circ$ " is defined by the following table:

$\circ$	e	a	b	c
e	e	a	b, c	b, c
a	a	b, c	e	e
b	b, c	e	a	a
c	b, c	e	a	a

(2.1)

Since the rows corresponding to the elements b and c are exactly the same, then $b \sim_o c$. Since it is obvious that the elements b and c occur together in each hyperproduct, we conclude that $\hat{b}_i = \hat{c}_i = \{b, c\}$, which finally gives that $\hat{b}_e = \hat{c}_e = \{b, c\}$. However, $\hat{e}_e = \{e\}$ and $\hat{a}_e = \{a\}$. Since there exists an element such that its equivalence class is not a singleton, the hypergroup is not reduced.
Let us construct the reduced form of the hypergroup H. According to Proposition 2.1, the obtained hypergroup will be reduced.
For ease of presentation we will illustrate a reduced form via Cayley table, too.

K	$\hat{e}_e$	$\hat{a}_e$	$\hat{b}_e$
$\hat{e}_e$	$\hat{e}_e$	$\hat{a}_e$	$\hat{b}_e$
$\hat{a}_e$	$\hat{a}_e$	$\hat{b}_e$	$\hat{e}_e$
$\hat{b}_e$	$\hat{b}_e$	$\hat{e}_e$	$\hat{e}_e$

(2.2)

It is easy to check that the hyperstructure $(K, \star)$, where $K = H/\sim_e$, is associative and that the reproducibility is satisfied. Thus, $(H/\sim_e, \star)$ is a hypergroup and it is obviously reduced.

As it is stated in Proposition 2.3, we can reconstruct a hyperoperation on H via the mapping $f: H \to H/\sim_e$. Indeed, $f^{-1}(f(a)\star f(b)) = f^{-1}(\hat{a}_e \star \hat{b}_e) = f^{-1}(\hat{e}_e) = e = a \circ b$. Similarly we can verify the statement for all other hyperproducts $x \circ y$, where $x, y \in H$.

In the following we give an example of a reduced hypergroup.

Example 2.3. Let $(H, \circ)$ be a hypergroup, where the hyperoperation "$\circ$" is defined by the following table.

$\circ$	a	b	c	d
a	a	a	a,b,c	a,b,d
b	a	a	a,b,c	a,b,d
c	a,b,c	a,b,c	a,b,c	c,d
d	a,b,d	a,b,d	c,d	a,b,d

(2.3)

One easily notices that $a \sim_o b$, because the lines (and columns) corresponding to a and b are exactly the same, thereby: $\hat{a}_o = \hat{b}_o = \{a, b\}$, while $\hat{c}_o = \{c\}$ and $\hat{d}_o = \{d\}$. But, on the other side, each element in H has equivalence class containing exactly one element with respect to the relation $\sim_i$, as well as with respect to the relation $\sim_e$, by consequence $(H, \circ)$ is reduced. Here, a reduced form $H/\sim_e$ is isomorphic to the hypergroup H.

In [24] Cristea et al. discussed about the regularity of the fundamental relations, proving that the operational equivalence and essential indistinguishability are regular, while the inseparability is not regular. Also, they proved that in general none of them is strongly regular. This means that the corresponding quotients modulo these equivalences are not classical structures, but hyperstructures.

If we consider Example 2.3, we can easily show that the relation $\sim_o$ is a regular relation, but not a strongly regular relation. Since $a \sim_o b$, then the regularity of the relation $\sim_o$ would imply that for every $u \in H$ and for every $x \in u \circ a$ there exists $y \in u \circ b$ such that $x \sim_o y$, and for every $x \in a \circ u$ there exists $y \in b \circ u$ such that $x \sim_o y$. Obviosly, for any $u \in H$, if $x \in u \circ a$ then $x \in u \circ b$, and it holds that $x \sim_o x$. Due to the commutativity of the hypergroup, the other relations from the regularity definition also easily follow.

However, the operational equivalence is not strongly regular. If we assume strongly regularity of the relation $\sim_o$ is , then $a \sim_o b$ implies that: $\forall x \in a \circ u, \forall y \in b \circ u$ there is $x \sim_o y$. Taking that $x = a, u = c$ and $y = c$, if follows that $a \sim_o c$, which is not satisfied.

Example 2.4. Let $H = \{a, b, c\}$ be the following hypergroup:

$\circ$	a	b	c
a	a	a, b, c	a, b, c
b	a, b, c	b, c	b, c
c	a, b, c	a, b, c	a, b, c

(2.4)

Notice that the elements b and c are essentialy indistinguishable, i.e., $b \sim_i c$. If we suppose that the relation $\sim_i$ is regular, we get: From $c \sim_i b$ it follows that for all $u \in H$ and for all $x \in c \circ u$ there exists $y \in b \circ u$ such that $x \sim_i y$. Taking that $u = b$, then for the element a which belongs to $c \circ b$, there does not exist an element y in $b \circ b$, such that $a \sim_i y$. Since $b \circ b = \{b, c\}$, indeed $a \nsim_i b$ and $a \nsim_i c$. Thus, the relation $\sim_i$ is not a regular relation.

Let us show, at the end, that in general, the relation $\sim_e$ is not a strongly regular relation.

Example 2.5. Let $H = \{a, b, c\}$ be the following hypergroup:

$\circ$	a	b	c
a	a, b	a, b	H
b	a, b	a, b	H
c	H	H	c

(2.5)

Is it easy to see that the relation $\sim_e$ is regular. Notice that the elements a and b are the only elements in the hypergroup such that $a \sim_e b$. Strongly regularity would imply that: For all $u \in H$ and $\forall x \in a \circ u, \forall y \in b \circ u : x \sim_e y$. Taking that $u = c$, the element a belongs to $a \circ c$, the element c belongs to $b \circ c$, but $a \nsim_e c$. Thus, the relation $\sim_e$ is not strongly regular.

2.2 Reducibility of hypergroups connected with the binary relations

In this section we present some results proved by Cristea and Stefanescu in [23]. They associated different hypergroupoids with binary relations defined on a set H. Also, they investigated the reducibility in hypergroups associated with the binary relation. The authors gave necessary and sufficient conditions for hypergroupoids in order to be reduced hypergroups. Further, they gave conditions such that hypergroups associated with the intersection, union and composition of relations are reduced.

Rosenberg has associated a partial hypergroupoid $H_\rho = (H, \circ)$ with a binary relation ρ defined on a set H, where for any $x, y \in H$, as [59]

$$x \circ x = L_x = \{z \in H : (x, z) \in \rho\}, \quad x \circ y = L_x \cup L_y.$$

Let ρ be a binary relation defined on a non-empty set H. Denote by L_x^ρ the set containing all elements z such that $x\rho z$, i.e., $L_x^\rho = \{z \in H : (x, z) \in \rho\}$. Similarly, $R_x^\rho = \{z \in H : (z, x) \in \rho\}$. If ρ and δ are two different binary relations on H, then:

$$L_x^{\rho\cap\delta} = \{z \in H : (x, z) \in \rho \cap \delta\} = L_x^\rho \cap L_x^\delta$$

$$R_x^{\rho\cap\delta} = \{z \in H : (z, x) \in \rho \cap \delta\} = R_x^\rho \cap R_x^\delta$$

$$L_x^{\rho\cup\delta} = \{z \in H : (x, z) \in \rho \cup \delta\} = L_x^\rho \cup L_x^\delta$$

$$R_x^{\rho\cup\delta} = \{z \in H : (z, x) \in \rho \cup \delta\} = R_x^\rho \cup R_x^\delta$$

$$L_x^{\rho\delta} = \{z \in H : (x, z) \in \rho\delta\} = \{z \in L_t^\delta : t \in L_x^\rho\}$$

$$R_x^{\rho\delta} = \{z \in H : (z, x) \in \rho\delta\} = \{z \in R_t^\rho : t \in R_x^\delta\}$$

If, for any $x \in H, L_x^\rho = L_x^\delta$ then $\rho = \delta$.

Proposition 2.4. *[23] The hypergroupoid H_ρ is reduced iff for all $x, y \in H$ such that $x \neq y$ either $L_x \neq L_y$ or $R_x \neq R_y$.*

Let H_ρ be a hypergroup associated with the binary relation ρ defined on H.

Proposition 2.5. [23] *If ρ is an equivalence on H, then the hypergroupoid H_ρ is a reduced hypergroup if and only if $\rho = \triangle_H = \{(x, x) : x \in H\}$.*

Example 2.6. Let $H = \mathbb{Z}$ and the relation ρ is given on the set H with:

$$x\rho y \quad \text{iff} \quad x \equiv y \quad (mod\, 5).$$

It is easy to check that the previous relation is an equivalence relation. But, according to Proposition 2.5, the hypergroup H_ρ associated with this relation is not reduced if $\rho \neq \{(x,x) : x \in H\}$. Indeed, if we set $5 \circ a = 10 \circ a$, where $a \in H$ then $L_5 \cup L_a = L_{10} \cup L_a$ which is obviously satisfied because the elements 5 and 10 belong to the same equivalence class with respect to the relation ρ, having the same remainder after dividing by 5. Similarly, the equality $a \circ 5 = a \circ 10$ is satisfied for any $a \in H$. Also, the elements 5 and 10 appear in the same hyperproducts because if $5 \in a \circ b = L_a \cup L_b$, i.e., $5\rho a$ or $5\rho b$, then certainly $10\rho a$ or $10\rho b$, which means that $10 \in a \circ b$. We conclude that the hypergroup H associated with the relation ρ is not reduced. If we choose the relation "is equal to" instead of the above defined equivalence relation, then it can be easily checked that the given hypergroup is reduced.

Proposition 2.6. [23] *If ρ is a non-symmetric quasiorder on H, then the hypergroup $(H_\rho, \circ)$ is reduced if and only if for any $x \neq y, L_x \neq L_y$.*

Example 2.7. Let $\leq$ be a quasiorder relation on $H = \mathbb{R}$. Then, according to the definition of a hypergroup $(H_\rho, \circ)$, the hyperproduct $x \circ x = L_x = \{z \in H : x\rho z\} = \{z \in H : x \leq z\}$, i.e., the hyperproduct $x \circ x$ contains all elements greater or equal to x. Here, for any two different elements x and y, the sets L_x and L_y are different, as well. Proposition 2.6 states that such hypergroup $(H_\rho, \circ)$ is a reduced hypergroup.
Let us prove it on this particular example. Let us assume that $x \circ a = y \circ a$, for all $a \in H$. If equality holds for any a, then it is obviosly satisfied for any $a \in H$, such that $a < x < y$. Since $x \circ y = L_x \cup L_y$, then the equality $x \circ a = y \circ a$ gives that $L_x = L_y$ which further implies that $x = y$. Thus, $x \circ a = y \circ a$ for any a implies that $x = y$, i.e., $\hat{x}_o = x$ for any $x \in R$. Hence, $\hat{x}_e = x$ for any $x \in H$, which finally gives that $(H_\rho, \circ)$ is a reduced hypergroup.

Proposition 2.7. [23] *If ρ is a reflexive, symmetric, non-transitive relation on H, such that $\rho^2 = H \times H$, then the hypergroup $(H_\rho, \circ)$ is reduced if and only if $L_x \neq L_y$ for all $x, y \in H$ such that $x \neq y$.*

Proposition 2.8. [23] *Let ρ and δ be two quasiorder relations on H. If the hypergroups H_ρ and H_δ are reduced then the hypergroup $H_{\rho\cap\delta}$ is reduced, too.*

Proposition 2.9. [23] *Let ρ and δ be two binary relations on H with full domain and full range such that $\rho^2 = \rho, \delta^2 = \delta$ and $\rho\delta = \delta\rho$. If the hypergroup $H_{\rho\delta}$ is reduced then H_ρ and H_δ are both reduced.*

In the following we present the new results related to the reducibility in hypergroups.

Proposition 2.10. *Any subhypergroup $(K, \circ)$ of a reduced hypergroup $(H, \circ)$ is a reduced hypergroup.*

Proof. Let a be the element from the set K. Since K is a subhypergroup, then $K \subseteq H$. Then the element a belongs to the set H, as well, and since $(H, \circ)$ is a reduced hypergroup, then $\hat{a}_e = \{a\}$ i.e., $(K, \circ)$ is a reduced hypergroup, too. $\square$

Remark 2.2. A subhypergroup of a non-reduced hypergroup can be reduced or not.

Example 2.8. Let the hypergroup $(H, \circ)$ be given by the following table

$\circ$	a	b	x	y
a	a	b	x, y	x, y
b	b	a	x, y	x, y
x	x, y	x, y	a, b	a, b
y	x, y	x, y	a, b	a, b

The hypergroup $(H, \circ)$ is non-reduced since $x \circ c = y \circ c$ for any $c \in H$ and x and y appear in the same hyperproducts. Thus, $x \sim_e y$ and consequently, $(H, \circ)$ is not a reduced hypergroup. Let us note with K the subset $\{a, b\}$ of the set H. Since $(K, \circ)$ is a hypergroup itself and $K \subset H$, then the hyperstructure $(K, \circ)$ is a subhypergroup of a hypergroup $(H, \circ)$. It is easy to see that $(K, \circ)$ is a reduced hypergroup.

In the following we show the interesting property of the reducibility, saying that the surjective homomorphism preserves reducibility.

Proposition 2.11. *Let ϕ be a good surjective homomorphism from the hypergroup $(R, +)$ to the hypergroup $(T, \oplus)$. If two elements are essential indistinguishable with respect to the hyperoperation $+$, then the images of the same elements through ϕ are in the essential indistinguishable relation with respect to the hyperoperation $\oplus$.*

Proof. Let x and y be elements from R such that $x + a = y + a$, where $a \in R$. This gives that $\{\phi(l) | l \in x + a\} = \{\phi(k) | k \in y + a\}$, so $\phi(x + a) = \phi(y + a)$. From here, $\phi(x) \oplus \phi(a) = \phi(y) \oplus \phi(a)$. Denote $\phi(a) = b$ and $\phi(x) = x_1, \phi(y) = y_1$. Thus, $x_1 \oplus b = y_1 \oplus b$. If the equality $x + a = y + a$ holds for every $a \in H$ then the last equality holds for all $b \in T$ since $\{\phi(a) | a \in R\} = T$. Assuming $a + x = y + x$ for all $a \in R$, similarly $\phi(a) \oplus \phi(x) = \phi(a) \oplus \phi(y)$ for all $a \in R$. Hence, if $x \sim_o^+ y$ then $\phi(x) \sim_o^\oplus \phi(y)$.

Let $x \sim_i^+ y$, i.e., $x \in a+b$ if and only if $y \in a+b$ for all $a, b \in R$. From this equivalence we get that $\phi(x) \in \{\phi(l) | l \in a + b\}$ if and only if $\phi(y) \in \{\phi(k) | k \in a + b\}$, so $\phi(x) \in$

$\phi(a+b)$ if and only if $\phi(y) \in \phi(a+b)$. Since ϕ is a homomorphism, $\phi(x) \in \phi(a) \oplus \phi(b)$ if and only if $\phi(y) \in \phi(a) \oplus \phi(b)$. Let $\phi(x) = x_1, \phi(y) = y_1$ and $\phi(a) = a_1, \phi(b) = b_1$. Since the mapping is surjective, $a_1 \oplus b_1$ covers the whole set T. Hence, $x_1 \in a_1 \oplus b_1$ is equivalent with $y_1 \in a_1 \oplus b_1$, for all $a_1, b_1 \in T$. Here, $x \sim_i^+ y$ implies $\phi(x) \sim_i^{\oplus} \phi(y)$. The definition of the essential indistinguishability relation, together with the above implications give the proof of the claim. $\square$

2.3 Reducibility in canonical hypergroups

In this section we study the reducibility of canonical hypergroups. After we investigate the reducibility for an arbitrary canonical hypergroup, we introduce a special class of canonical hypergroups, so called i.p.s. hypergroups. We present here some important properties of these hypergroups, neccesary for the study of their fuzzy reducibility.

Theorem 2.1. *Any canonical hypergroup is a reduced hypergroup.*

Proof. Since any canonical hypergroup has a scalar identity 0 such that $0 \circ x = x \circ 0 = x$ for any $x \in H$, then if we set $a \circ x = a \circ y$ for any $a \in H$, by taking $a = 0$ we get: $0 \circ x = 0 \circ y$, which implies that $x = y$. Thus, $\hat{x}_o = \{x\}$ for all $x \in H$, so obviously $\hat{x}_e = \{x\}$ for any $x \in H$. Hence, H is a reduced hypergroup. $\square$

Remark 2.3. In the previous theorem, since an arbitrary element has singleton equivalence class with respect to the operational equivalence, then obviosly, it has a singleton equivalence class with respect to the essential indistinguishability, but even more, it holds that $\hat{x}_i = \{x\}$ for all $x \in H$, since two elements does not appear in the same hyperproducts $a \circ b$, where $a, b \in H$. Since $x \in x \circ 0$, the element y belongs to the same hyperproduct just if $y = x$, i.e., $x \in a \circ b$ if and only if $y \in a \circ b$ holds only if $x = y$.

Example 2.9. Let $S = \{-1, 0, 1\}$, and the hyperoperation $\circ$ is given by the following table

$\circ$	-1	0	1
-1	-1	-1	H
0	-1	0	1
1	H	1	1

The hypergroup $(H, \circ)$ is a canonical hypergroup. Indeed, 0 is a scalar identity, since for all $x \in H, x \circ 0 = 0 \circ x = x$. Also, since $0 \in 0 \circ 0$ then $0^{-1} = 0$. Similarly, $0 \in 1 \circ (-1)$ and $0 \in -1 \circ 1$, and consequently, $1^{-1} = -1$ and $(-1)^{-1} = 1$. Thus, every element has a unique inverse. At the and, let us check the last condition of Definition 1.9. If

$0 \in 1 \circ (-1)$, then -1 belongs to the sets $1^{-1} \circ 0$ and $0 \circ (-1)^{-1}$. Similarly, the condition can be checked for the other hyperproducts.
Since the rows in the table are distinct, we conclude that $\hat{x}_o = \{x\}$, for $x \in \{-1, 0, 1\}$. Hence $\hat{x}_e = \{x\}$, for any x in H, which gives that the given hypergroup is reduced.

The following proposition states that the canonical hypergroup modulo canonical subhypergroup is a reduced hypergroup.

Proposition 2.12. *Let $(H, +)$ be a canonical hypergroup and N be an arbitrary canonical subhypergroup of H. Then the quotient $H \setminus N$ is a reduced hypergroup.*

Proof. The Proposition is the direct consequence of Proposition 1.1 and Theorem 2.1. ☐

Now we will introduce a class of canonical hypergroups, called i.p.s. hypergroups.

An i.p.s. hypergroup is a canonical hypergroup with partial scalar identities [12]. Its name, given by Corsini [12] comes from the Italian language, and the abbreviation "i.p.s." is derived from the "identità parziale scalare", which translated into English, means partial scalar identity. We have to keep in mind that the notion of a partial scalar identity and the notion of a identity in a hypergroup $(H, \circ)$ must not be confused. Recall that an element $x \in H$ is called a *scalar*, if $|x \circ y| = |y \circ x| = 1$, for any $y \in H$. An element $e \in H$ is called *partial identity* of H if it is a *left identity* (i.e., there exists $x \in H$ such that $x \in e \circ x$) or a *right identity* (i.e., there exists $y \in H$ such that $y \in y \circ e$) [12]. Denote the set of all partial identities of H by I_p. Besides, for a given element $x \in H$, a *partial identity of x* is an element $u \in H$ such that $x \in x \circ u \cup u \circ x$. The element $u \in H$ is a *partial scalar identity of x* if $x \in x \circ u$ implies that $x = x \circ u$ and whenever $x \in u \circ x$ it follows that $x = u \circ x$. For any element x, $I_p(x)$ denotes the set of all partial identities of x, $I_{ps}(x)$ denotes the set of all partial scalar identities of x, while $Sc(H)$ denotes the set of all scalars of H. It is easy to see that the intersection of the sets $I_p(x)$ and $Sc(H)$ gives the set $I_{ps}(x)$.

Remark 2.4. Regarding the expression "partial identity" we have to pay attention on the term "partial", that does not mean "left or right" (identity). An element u is a *partial identity* is equivalent with the fact that u behaves *partially* as an identity with respect to an element x. Thus, u is not a left/right (i.e., partial) identity for the hypergroup H. Besides, an i.p.s. hypergroup is a commutative hypergroup, so the concept of partial intended as left/right element satisfying a property (i.e. left/right unit) has no sense. Therefore, we observe that an element u has the property of being *partial* identity for x means that that it has a similar behaviour as an identity but only with respect to x (and not all the elements), so a *partial* role of being identity.

Let us recall now the definition of an i.p.s. hypergroup. All finite i.p.s. hypergroups of order less than 9 have been determined by Corsini [10, 11, 12].

Definition 2.3. [12] *A hypergroup $(H, \circ)$ is called i.p.s. hypergroup, if it satisfies the following conditions.*

1. *It is canonical, i.e.,*
   - *it is commutative;*
   - *it has a scalar identity* 0 *such that* $0 \circ x = x$, *for any* $x \in H$;
   - *every element* $x \in H$ *has a unique inverse* $x^{-1} \in H$, *that is* $0 \in x \circ x^{-1}$;
   - *it is reversible, so* $y \in a \circ x \Longrightarrow x \in a^{-1} \circ y$, *for any* $a, x, y \in H$.
2. *It satisfies the relation: for any* $a, x \in H$, *if* $x \in a \circ x$, *then* $a \circ x = x$.

The most useful properties of i.p.s. hypergroups are gathered in the following result.

Proposition 2.13. [12] *Let $(H, \circ)$ be an i.p.s. hypergroup.*

1. *For any* $x \in H$, *the set* $x \circ x^{-1}$ *is a subhypergroup of* H.
2. *For any* $x \in H \setminus \{0\}$, *we have: or* $x \in Sc(H)$, *or there exists* $u \in Sc(H) \setminus \{0\}$ *such that* $u \in x \circ x^{-1}$. *Moreover* $|Sc(H)| \geq 2$.
3. *If* $x \in Sc(H)$, *then* $I_{ps}(x)$ *contains just* 0.
   If $x \notin Sc(H)$, *then* $I_{ps}(x) \subset Sc(H) \cap x \circ x^{-1}$ *and therefore* $|I_{ps}(x)| \geq 2$.

Proposition 2.14. *Let $(H, \circ)$ be an i.p.s. hypergroup. For any scalar $u \in H$ and for any element $x \in H$, there exists a unique $y \in H$ such that $u \in x \circ y$.*

Proof. The existence immediately follows from reproducibility. For proving the unicity, assume that there exist $y_1, y_2 \in H, y_1 \neq y_2$ such that $u \in x \circ y_1 \cap x \circ y_2$. Then, by reversibility, it follows that $y_1, y_2 \in x^{-1} \circ u$. Since u is a scalar element, we get $|x^{-1} \circ u| = 1$ and then $y_1 = y_2 = x^{-1} \circ u$. □

Example 2.10. [12] Let us consider the following i.p.s. hypergroup $(H, \circ)$.

H	0	1	2	3
0	0	1	2	3
1	1	2	0, 3	1
2	2	0, 3	1	2
3	3	1	2	0

Here we can notice that 0 is the only one identity of H. In addition, $Sc(H) = \{0, 3\}$, and $0 \in 0 \circ u$ only for $u = 0$, so $I_{ps}(0) = \{0\}$. Also, only for $u = 0$ there is $3 \in 3 \circ u$, thus $I_{ps}(3) = \{0\}$. (In general, if $x \in Sc(H)$ then $I_{ps}(x) = \{0\}$, according with Proposition 2.13.) Similarly, one gets $I_p(1) = I_p(2) = \{0, 3\}$ and since $Sc(H) = \{0, 3\}$, it follows that $I_{ps}(1) = I_{ps}(2) = \{0, 3\}$.

Note that, in an i.p.s. hypergroup, the Jantosciak fundamental relations have a particular meaning, in the sense that, for any two elements there is

$$a \sim_o b \Longleftrightarrow a \sim_i b \Longleftrightarrow a \sim_e b \Longleftrightarrow a = b. \quad [22]$$

By consequence, one obtains the following result.

Theorem 2.2. *Any i.p.s. hypergroup is reduced.*

Proof. This is the direct consequence of the Theorem 2.1. □

2.4 Reducibility in some cyclic hypergroups

Cyclic hypergroups have been introduced by De Salvo and Freni [36] and Vougiouklis [66] independently. The notion of cyclicity is well known since it is an important concept in theory of algebraic structures. The hypergroup is called cyclic if we can obtain whole hypergroup applying a hyperoperation on a specific element which represents a generator of a hypergroup. Corsini did a synthesis of two approaches in his book [13] and gave definitions using unambiguous terminology. After we recall the definitions we will examine reducibility for certain types of cyclic hypergroups and present examples of some (non) reduced cyclic hypergroups.

Definition 2.4. [13] *A hypergroup H is called cyclic with a generator x if $\phi_H(H)$ is a cyclic group generated from $\phi_H(x)$, where ϕ_H is a canonical projection.*

Definition 2.5. [13] *A semihypergroup is called cyclic if there exists $h \in H$ such that $\forall x \in H \quad \exists n \in \mathbb{N}$ such that $x \in h^n$. We call h the s-generator of H. A hypergroup is called s-cyclic if it is a cyclic semihypergroup.*

Definition 2.6. [66] *A hypergroup $(H, \circ)$ is called a single-power cyclic hypergroup if there exists $h \in H$ and $s \in \mathbb{N}$ such that $H = h \cup h^2 \cup \cdots h^s \cup \cdots$ and $h \cup h^1 \cup h^2 \cup \cdots h^{m-1} \subset h^m$ for every $m \in \mathbb{N}$. The smallest power s for which formula $H = h \cup h^2 \cup \cdots h^s \cup \cdots$ is valid is called a period of h.*

Proposition 2.15. *The only single-power, non-reduced cyclic hypergroup of order two is the total hypergroup.*

Proof. Let H be a hypergroup of order two, where $H = \{a, b\}$. If we suppose that $a \circ x = b \circ x$, and $x \circ a = x \circ b$ for all $x \in H$ and if $a^2 = H$ then it holds that $a \circ a = b \circ a = H$. Also, $b \circ a = b \circ b$ which implies that $b \circ b = H$. Similarly, $a \circ b = H$. Thus, H is a total hypergroup. □

Example 2.11. Let H be a hypergroup given by the following table

$\circ$	a	b
a	H	b
b	b	a

(2.6)

The hypergroup $(H, \circ)$ is a single-power cyclic hypergroup and it is easy to see that it is a reduced hypergroup.

Proposition 2.16. *Let H be a commutative single-power cyclic hypergroup of period 2, such that all its elements are generators, with $|H| = 3$. Then the hypergroup H is not reduced only if it is a total hypergroup.*

Proof. Let $H = \{a, b, c\}$ and $a^2 = b^2 = c^2 = H$. In order to be a non-reduced hypergroup, the equivalence class of at least one element has to be a non-singleton set. Let us suppose that $\hat{a}_e = \hat{b}_e = \{a, b\}$. This means that $a \sim_o b$, which obviosly implies that $a \circ a = a \circ b = H$. Due to the commutativity there is $b \circ a = H$. In order to make elements a and b be operationally equivalent, it must be valid that $a \circ c = b \circ c$. Now we will consider all possible options for the hyperproduct $a \circ c = b \circ c$.

If $a \circ c = b \circ c = a$, then due to the associativity it is valid that $(a \circ c) \circ c = a \circ (c \circ c)$. Since $a \circ (c \circ c) = a \circ H = H$, then $(a \circ c) \circ c = a \circ c = H$ which contradicts with the assumption.

Similarly, taking that $a \circ c = b \circ c = b$, and using that $(b \circ c) \circ c = b \circ (c \circ c) = b \circ H = H$ we get that $a \circ c = H$, which is false.

At the end, if $a \circ c = b \circ c = c$, using the associativity rule for $b \circ (b \circ c)$, we again get a contradiction.

The only remaining options for this hyperproduct are the sets: $\{a, b\}, \{a, c\}, \{b, c\}$. We won't consider the last two, because then it would hold that $a \nsim_i b$, which contradicts with the assumption that $a \sim_e b$. Hence, the only possible option is that $a \circ c = b \circ c = \{a, b\}$, but such a structure is not a hypergroup since the associativity rule is not satisfied. Namely, $(a \circ c) \circ c = a \circ (c \circ c)$, but the left side of equality is equal to $\{a, b\}$, while the right side is equal to H. The proof is analogous if we assume that $a \sim_e c$ or $b \sim_e c$. We conclude that the only hypegroup which satisfies the conditions of the Proposition and it is non-reduced, is the total hypergroup. $\square$

Notice that in the case when $|H| = 4$ the previous Proposition doesn't hold.

Example 2.12. Let the hypergroup is given by the following table.

$\circ$	a	b	c	d
a	H	H	d	H
b	H	H	d	H
c	d	d	H	H
d	H	H	H	H

(2.7)

The hyperstructure given by the above table is a commutative single-power cyclic hypergroup. Also, every element of H is a generator with the period 2. Notice that the elements a and b appear in the same hyperproducts, which gives that $a \sim_i b$. From the table we can see that $a \circ x = b \circ x$ and $x \circ a = x \circ b$, for any $x \in H$. Hence, $x \sim_o y$. Therefore the hypegroup $(H, \circ)$ is not a reduced hypergroup.

Now we will present an example of an infinite single-power hypergroup and study its reducibility.

Example 2.13. Let I be an open interval $I = (0, 1)$ and let the hyperoperation be given by: $a \star b = [a \cdot b)_{\leq} = \{x \in I : a \cdot b \leq x\}$. In [55] it has been proved that the structure $(I, \star)$ is a single power cyclic hypergroup with an infinite period for an arbitrary $a \in I$. Let us prove that the hypergroup is reduced.
Let a and a_1 be elements from I such that $a \circ b = a_1 \circ b$ for all b in I. Then $[a \cdot b)_{\leq} = [a_1 \cdot b)_{\leq}$, i.e., $\{x : x \geq ab\} = \{x : x \geq a_1 b\}$, which is fulfilled just in the case when $a \cdot b = a_1 \cdot b$. Thus, $a = a_1$. Hence, for all $a \in I$ it holds that $\hat{a}_o = \{a\}$, and thus $\hat{a}_e = \{a\}$, for all $a \in I$. Therefore, $(I, \star)$ is a reduced hypergroup.

In the following, we will show examples of hypergroups which are join spaces and are reduced hypergroups.

Example 2.14. Let ρ be a reflexive and symmetric relation on H. Let us consider the hyperoperation on H given with:

$$\forall (x, y) \in H^2, x \circ x = L_x, x \circ y = L_x \cup L_y, \text{where} \quad L_x = \{z \in H : (x, z) \in \rho\}.$$

In [48], L. Loreanu has proved that the hyperstructure $(H, \circ)$ is a join space. As we have already seen at the beginning of the chapter, this hyperoperation defined above was introduced by Rosenberg. Let the relation ρ be given on the set $H = \{x, y, z\}$ with:
$\rho = \{(x, x), (y, y), (z, z), (x, z), (z, x), (y, z), (z, y)\}$.
It is easy to check that the relation ρ is reflexive and symmetric. Using the definition of the hyperoperation "$\circ$" we get that $x \circ y = x \circ z = y \circ z = H$, while $x \circ x = \{x, z\}$, $y \circ y = \{y, z\}$ and $z \circ z = H$. The hypergroup $(H, \circ)$ is a reduced hypergroup, since we notice that arbitrary two elements from H are not operationally equivalent, nor inseparable.

Example 2.15. Let V be a vector space over an ordered field F. If $a, b \in V$ we can define: $a \circ b = \{\lambda a + \mu b : \lambda > 0, \mu > 0, \lambda + \mu = 1\}$, then $(V, \circ)$ is a join space, called an affine join space over F [32].
In the following we prove that $(H, \circ)$ is a reduced hypergroup.
Let a and b be two arbitrary elements from V such that $a \sim_o b$, i.e., $a \circ x = b \circ x$ for all $x \in V$. Using the definition of the hyperoperation " $\circ$ ", and taking that $x = a$ we get: $a \circ a = b \circ a$. Thus, $\{\lambda a + \mu a : \lambda > 0, \mu > 0, \lambda + \mu = 1\} = \{\lambda b + \mu a : \lambda > 0, \mu > 0, \lambda + \mu = 1\}$. Since the first set contains just the point a, it will be equal to the segment $[a, b]$ just in the case when $a = b$. Thus, $a \sim_o b$ implies $a = b$. From here, $\hat{a}_e = \{a\}$ for all $a \in V$, i.e., H is a reduced hypergroup.

2.5 Reducibility in complete hypergroups

In this section we recall a very important class of hypergroups, so called complete hypergroups. We give the definition of a complete hypergroup and we also describe a way how to construct different complete hypergroups. In order to study the reducibility in complete hypergoups, we introduce a certain equivalence relation in order to identify elements which are in the same equivalence class with respect to the relation $\sim_e$. Let $(H, \circ)$ be a proper complete hypergroup (i.e. H is not a group). Define now on H the equivalence " $\sim$ " by:

$$x \sim y \Longleftrightarrow \exists g \in G \text{ such that } x, y \in A_g. \tag{2.8}$$

Proposition 2.17. *On a proper complete hypergroup $(H, \circ)$, the equivalence $\sim$ in (2.8) is a representation of the essentially indistinguishability equivalence $\sim_e$.*

Proof. By Theorem 1.11, one notices that, for any element $x \in H$, there exists a unique $g \in G$, namely g_x, such that $x \in A_{g_x}$. First, suppose that $x \sim y$, i.e., there exists $g_x = g_y \in G$ such that $x, y \in A_{g_x}$. For any arbitrary element $a \in H$, we can say

that $a \in A_{g_a}$, with $g_a \in G$, and by the definition of the hyperproduct in the complete hypergroup $(H, \circ)$, there is $x \circ a = A_{g_x g_a} = A_{g_y g_a} = y \circ a$, (and similarly, $a \circ x = a \circ y$,) implying that $x \sim_o y$ (i.e., x and y are operationally equivalent.) Secondly, for any $x \in a \circ b = A_{g_a g_b} \cap A_{g_x}$, it follows that $g_a g_b = g_x$; but $g_x = g_y$, so $y \in A_{g_a g_b} = a \circ b$. Thereby, x belongs to the set $a \circ b$ if and only if y belongs to the same set, which means that $x \sim_i y$ (i.e. x and y are inseparable). We have proved that $\sim \subseteq \sim_e$.
Conversely, let us suppose that $x \sim_e y$. Since x and y are inseparable, i.e., $x \in a \circ b$ if and only if $y \in a \circ b$, we may write $x, y \in A_{g_a g_b}$. Therefore there exists $g_x = g_a \cdot g_b \in G$ such that $x, y \in A_{g_x}$, so $x \sim y$. $\square$

Example 2.16. Considering Example 1.5, we notice that the equivalence classes of the elements of H with respect to the equivalence $\sim$ defined in (2.8) are: $\hat{e} = \{e\}$, $\hat{a_1} = \hat{a_2} = \hat{a_3} = \{a_1, a_2, a_3\}$. If we consider $\sim_e$, we again get the same equivalences classes: $\hat{e}_e = \{e\}$, $\hat{a_1} = \{a_1, a_2, a_3\} = \hat{a_2}_e = \hat{a_3}_e$.

Theorem 2.3. *Any proper complete hypergroup is not reduced.*

Proof. Let $(H, \circ)$ be a proper complete hypergroup (meaning that it does not coincides with a group). Then there exists at least one element g in G such that $|A_g| \geq 2$. From here we conclude that there exists elements a and b, in H with $a \neq b$, such that $a \sim b$. Thence, $a \sim_e b$ which directly proves non-reducibility of $(H, \circ)$. $\square$

The following example is an example of a complete hypergroup which is generated by the non-commutative group of quaternions.

Example 2.17. Let $(H, \circ)$ be a hypergroup represented by the following Cayley table:

H	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}
a_1	a_1, a_2	a_1, a_2	a_3, a_4	a_3, a_4	a_5	a_6, a_7, a_8	a_6, a_7, a_8	a_6, a_7, a_8	a_9	a_{10}	a_{11}, a_{12}	a_{11}, a_{12}	a_{13}
a_2	a_1, a_2	a_1, a_2	a_3, a_4	a_3, a_4	a_5	a_6, a_7, a_8	a_6, a_7, a_8	a_6, a_7, a_8	a_9	a_{10}	a_{11}, a_{12}	a_{11}, a_{12}	a_{13}
a_3	a_3, a_4	a_3, a_4	a_1, a_2	a_1, a_2	a_6, a_7, a_8	a_5	a_5	a_5	a_{10}	a_9	a_{13}	a_{13}	a_{11}, a_{12}
a_4	a_3, a_4	a_3, a_4	a_1, a_2	a_1, a_2	a_6, a_7, a_8	a_5	a_5	a_5	a_{10}	a_9	a_{13}	a_{13}	a_{11}, a_{12}
a_5	a_5	a_5	a_6, a_7, a_8	a_6, a_7, a_8	a_3, a_4	a_1, a_2	a_1, a_2	a_1, a_2	a_{11}, a_{12}	a_{13}	a_{10}	a_{10}	a_9
a_6	a_6, a_7, a_8	a_6, a_7, a_8	a_5	a_5	a_1, a_2	a_3, a_4	a_3, a_4	a_3, a_4	a_{13}	a_{11}, a_{12}	a_9	a_9	a_{10}
a_7	a_6, a_7, a_8	a_6, a_7, a_8	a_5	a_5	a_1, a_2	a_3, a_4	a_3, a_4	a_3, a_4	a_{13}	a_{11}, a_{12}	a_9	a_9	a_{10}
a_8	a_6, a_7, a_8	a_6, a_7, a_8	a_5	a_5	a_1, a_2	a_3, a_4	a_3, a_4	a_3, a_4	a_{13}	a_{11}, a_{12}	a_9	a_9	a_{10}
a_9	a_9	a_9	a_{10}	a_{10}	a_{13}	a_{11}, a_{12}	a_{11}, a_{12}	a_{11}, a_{12}	a_3, a_4	a_1, a_2	a_5	a_5	a_6, a_7, a_8
a_{10}	a_{10}	a_{10}	a_9	a_9	a_{11}, a_{12}	a_{13}	a_{13}	a_{13}	a_1, a_2	a_3, a_4	a_6, a_7, a_8	a_6, a_7, a_8	a_5
a_{11}	a_{11}, a_{12}	a_{11}, a_{12}	a_{13}	a_{13}	a_{13}	a_{10}	a_{10}	a_{10}	a_6, a_7, a_8	a_5	a_3, a_4	a_3, a_4	a_1, a_2
a_{12}	a_{11}, a_{12}	a_{11}, a_{12}	a_{13}	a_{13}	a_{13}	a_{10}	a_{10}	a_{10}	a_6, a_7, a_8	a_5	a_3, a_4	a_3, a_4	a_1, a_2
a_{13}	a_{13}	a_{13}	a_{11}, a_{12}	a_{11}, a_{12}	a_{10}	a_9	a_9	a_9	a_5	a_6, a_7, a_8	a_1, a_2	a_1, a_2	a_3, a_4

(2.9)

The hypergroup $(H, \circ)$ is complete, where the group $G = \mathbb{Q}_8 = \{\mp 1, \mp i, \mp j, \mp k\}$. The hypergroup H can be partitioned into disjoint sets: $A_0 = \{e\}, A_1 = \{a_1, a_2\}, A_{-1} = \{a_3, a_4\}, A_i = \{a_5\}, A_{-i} = \{a_6, a_7, a_8\}, A_j = \{a_9\}, A_{-j} = \{a_{10}\}, A_k = \{a_{11}, a_{12}\}, A_{-k} = \{a_{13}\}$, and $H = \bigcup_{g \in G} A_g$.

Remark 2.5. The conjugable subhypergroup of a complete hypergroup is not a reduced hypergroup. It is known that every conjugable subhypergroup of a hypergroup is a complete part [35]. According to Theorem 2.3, two arbitrary elements from the complete part are in the same equivalence class with respect to the essential indistinguishability relation, hence any conjugable subhypergroup of a complete hypergroup (which must be a complete part) is not a reduced hypergroup, too.

2.6 The reducibility in Corsini hypergroups

In this section we study the reducibility in Corsini hypergroups. We determine necessary and sufficient conditions for Corsini hypergroups to be reduced and study the reducibility in the productional hypergroups containing Corsini hypergroups.

Proposition 2.18. *Let $(H, \circ)$ be a Corsini hypergroup. If there exist some different elements x, y in H such that $x \circ x = y \circ y$, then the hypergroup $(H, \circ)$ is not reduced.*

Proof. Let x, y be arbitrary elements in H such that $x \neq y$ and $x \circ x = y \circ y$. It is easy to see that $x \circ a = y \circ a$, for any $a \in H$, since $x \circ a = x \circ x \cup a \circ a = y \circ y \cup a \circ a = y \circ a$. Using the commutativity, we obtain that $a \circ x = a \circ y$, for any $a \in H$. Hence, $x \sim_o y$. Let $x \in c \circ d$, with $x, c, d \in H$. Then $x \in c \circ c \cup d \circ d$, which implies that $x \in c \circ c$ or $x \in d \circ d$. Since $(H, \circ)$ is a Corsini hypergroup, the previous implication gives $c \in x \circ x$ or $d \in x \circ x$ and $c \in y \circ y$ or $d \in y \circ y$. Using the same property, we conclude that $y \in c \circ d$. Similarly, one proves the converse implication. Therefore, $x \sim_i y$. Hence, the hypergroup $(H, \circ)$ is not reduced. □

As a consequence of Proposition 2.18, we obtain the following results. It gives necessary and sufficient condition for the Corsini hypergroup to be reduced.

Proposition 2.19. *A Corsini hypergroup $(H, \circ)$ with at least two different elements is reduced if and only if $x \circ x \neq y \circ y$, for all $x, y \in H$.*

Proof. The contraposition of Proposition 2.18 directly gives the first direction. Suppose now that $x \circ x \neq y \circ y$, for all $x, y \in H$. Take two arbitrary elements $x \neq y$ from H. We will prove that $x \circ a = y \circ a$, for all $a \in H$, just in case when $x = y$. Assume that $x \circ a = y \circ a$, for all $a \in H$. From here, we have $x \circ x = y \circ x$, which gives $x \circ x = y \circ y \cup x \circ x$. The last equality is possible only if $y \circ y \subseteq x \circ x$. Similarly, since $x \circ y = y \circ y$, it follows the other inclusion $x \circ x \subseteq y \circ y$. Therefore, $x \circ a = y \circ a$ is equivalent with $x \circ x = y \circ y$, which contradicts the hypothesis. Hence, two arbitrary elements x and $y, x \neq y$ are not operationally equivalent, thus $\hat{x}_e = \{x\}$ for all $x \in H$, meaning that H is a reduced hypergroup. □

Proposition 2.20. *Any B-hypergroup is reduced.*

Proof. This immediately follows from Proposition 2.19, since in a B-hypergroup there is $x \circ x = \{x\}$, for all elements x. □

The following example shows Cayley table of a $B-$ hypergroup $(H, \circ)$, where $|H| = 3$.

Example 2.18.

$\circ$	x	y	z
x	x	x, y	x, z
y	x, y	y	y, z
z	x, z	y, z	z

In the following example we present a reduced Corsini hypergroup, which is not a B-hypergroup.

Example 2.19. On the set $H = \{a, b, c\}$ define the hyperoperation "$\circ$" by the following table:

$\circ$	a	b	c
a	H	H	H
b	H	a, b	H
c	H	H	a, c

Since all the rows in the table are different, it follows that $\hat{x}_o = \{x\}$ for any $x \in H$, which clearly implies the reducibility of the hypergroup.

The following theorem determines whether the direct product of hypergroups is reduced, or not.

Theorem 2.4 ([23]). *The hypergroup $(H \times H, \bigotimes)$ is reduced if and only if the hypergroups $(H, \circ_1)$ and $(H, \circ_2)$ are reduced.*

Example 2.20. Let $H = \{a, b\}$ and the hyperoperoperations $\circ_1$ and $\circ_2$ are given with the tables

$\circ_1$	a	b
a	H	H
b	H	H

$\circ_2$	a	b
a	a	H
b	H	b

The hyperproduct of the hypergroups $(H, \circ_1)$ and $(H, \circ_2)$ is the productional hypergroup $(H \times H, \bigotimes)$ given by the following table

$\circ_1 \times \circ_2$	(a,a)	(a,b)	(b,a)	(b,b)
(a,a)	$\{(a,a),(b,a)\}$	$H \times H$	$\{(a,a),(b,a)\}$	$H \times H$
(a,b)	$H \times H$	$\{(a,b),(b,b)\}$	$H \times H$	$\{(a,b),(b,b)\}$
(b,a)	$\{(a,a),(b,a)\}$	$H \times H$	$\{(a,a),(b,a)\}$	$H \times H$
(b,b)	$H \times H$	$\{(a,b),(b,b)\}$	$H \times H$	$\{(a,b),(b,b)\}$

Since the total hypergroup is not-reduced, according to Theorem 2.4, the productional hypergroup is not reduced, too. Indeed, $(a,a) \sim_e (b,a)$ and $(a,b) \sim_e (b,b)$ which implies the non-reducibility of $(H \times H, \circ_1 \times \circ_2)$.

Proposition 2.21. *The direct product of B-hypergoups is reduced.*

Proof. Since any B-hypergroup is reduced, this is a direct corollary of Theorem 2.4. $\square$

Chapter 3

Fuzzy reducibility in hypergroups

The following chapter is dedicated to the study of the fuzzy reducibility. Here we consider crisp hypergroups endowed with a fuzzy set and investigate their reducibility.

As already mentioned in the introductory part of this thesis, the extension of the concept of reducibility to the fuzzy case can be performed on a crisp hypergroup endowed with a fuzzy set, by defining, similarly to the classical case, three equivalences as follows.

Definition 3.1. [22] *In a crisp hypergroup $(H, \circ)$ endowed with a fuzzy set μ, we define the following equivalences:*

1. *x and y are fuzzy operationally equivalent and write $x \sim_{fo} y$ if, for any $a \in H$, $\mu(x \circ a) = \mu(y \circ a)$ and $\mu(a \circ x) = \mu(a \circ y)$;*
2. *x and y are fuzzy inseparable and write $x \sim_{fi} y$ if $\mu(x) \in \mu(a \circ b) \Longleftrightarrow \mu(y) \in \mu(a \circ b)$, for $a, b \in H$;*
3. *x and y are fuzzy essentially indistinguishable and write $x \sim_{fe} y$, if they are fuzzy operationally equivalent and fuzzy inseparable.*

Definition 3.2. [22] *The crisp hypergroup $(H, \circ)$ is a fuzzy reduced hypergroup if and only if the equivalence class of each element in H with respect to the fuzzy essentially indistinguishable relation is a singleton, i.e.,*

$$\textit{for all } \ x \in H, \hat{x}_{fe} = \{x\}.$$

Notice that the notion of fuzzy reducibility of a hypergroup is strictly connected with the definition of the involved fuzzy set.

Remark 3.1. [22] It is easy to see that, in any hypergroup H endowed with an arbitrary fuzzy set μ, the following implication holds: for any $a, b \in H$,

$$a \sim_o b \Rightarrow a \sim_{fo} b.$$

Remark 3.2. (i) Let us first clarify in detail the meaning of $\mu\,(a \circ b)$, for any $a, b \in H$ and any arbitrary fuzzy set μ defined on H. Generally, the hyperproduct $a \circ b$ is a subset of H, so $\mu\,(a \circ b)$ is the direct image of this subset through the fuzzy set μ, i.e., $\mu(a \circ b) = \{\mu\,(x) \mid x \in a \circ b\}$.

It is important to emphasize the following relations. If the hyperproduct $a \circ b$ is a singleton, i.e., $a \circ b = \{c\}$, then $\mu\,(a \circ b)$ is a set which contains only the real number $\mu\,(c)$. Thereby, we can write $\mu\,(c) \in \mu\,(a \circ b)$, nevertheless we cannot write $\mu(c) = \mu(a \circ b)$, because the first member is a real number, while the second one is a set containing the real number $\mu\,(c)$.

Moreover, if $a \in x \circ y$, then, obviosly $\mu\,(a) \in \mu\,(x \circ y)$, but the conversely doesn't hold, because it may happen that $\mu\,(a) \in \mu\,(x' \circ y')$ for $a \notin x' \circ y'$.

(ii) Generally, $a \sim_i b \nRightarrow a \sim_{fi} b$, as we can see in the next example. Indeed, if $a \sim_i b$ then $a \in x \circ y$ if and only if $b \in x \circ y$. But it may happen that $\mu\,(a) \in \mu\,(x' \circ y')$ with $a \notin x' \circ y'$, so also $b \notin x' \circ y'$. Also, if $\mu(a) \neq \mu(b)$, then $\mu\,(b) \notin \mu\,(x' \circ y')$, thus $a \not\sim_i b$.

(iii) Finally, it is easy to conclude that $\mu\,(a) = \mu\,(b)$ implies $a \sim_{fi} b$.

The following example justifies all the above mentioned assertions.

Example 3.1. [19] Let $(H, \circ)$ be a hypergroup represented by the following commutative Cayley table:

$\circ$	e	a_1	a_2	a_3
e	e	a_1	a_2, a_3	a_2, a_3
a_1		a_2, a_3	e	e
a_2			a_1	a_1
a_3				a_1

(3.1)

One notices immediately that $a_2 \sim_i a_3$, while $a_1 \not\sim_i a_2$.
$a)$ Define now on H the fuzzy set μ as follows: $\mu\,(e) = 1, \mu\,(a_1) = \mu\,(a_2) = 0.3, \mu\,(a_3) = 0.5$. Since $\mu\,(a_1) = \mu\,(a_2)$, it follows that $a_1 \sim_{fi} a_2$. Moreover, since $e \circ a_1 = \{a_1\}$, we have

$$\mu\,(e \circ a_1) = \{\mu\,(a_1)\} = \{0.3\} \ni \mu\,(a_2)\,,$$

while it is clear that $\mu\,(a_3) \notin \mu\,(e \circ a_1)$, so $a_2 \not\sim_{fi} a_3$.

b) If we define on H the fuzzy set μ by taking $\mu(e) = \mu(a_1) = 1, \mu(a_2) = \mu(a_3) = \frac{1}{3}$, it follows that $e \sim_{fi} a_1$ and $a_2 \sim_{fi} a_3$.

As we have already underlined, it is clear that the equivalences $\sim_{fo}$, $\sim_{fi}$, and $\sim_{fe}$ are strictly related with the definition of the fuzzy set considered on the hypergroup.

In the following we present the example of an infinite hypergroup and study its fuzzy reducibility.

Example 3.2. Consider the partially ordered group $(\mathbb{Z}, +, \leq)$ with the usual addition and orderings of integers. Define on $\mathbb{Z}$ the hyperoperation $a * b = \{x \in \mathbb{Z} \mid a + b \leq x\}$. Then $(\mathbb{Z}, *)$ is a hypergroup [55]. Define now on $\mathbb{Z}$ the fuzzy set μ as follows: $\mu(0) = 0$ and $\mu(x) = \frac{1}{|x|}$, for any $x \neq 0$. We obtain $\hat{x}_{fo} = \{x\}$, for any $x \in \mathbb{Z}$, therefore $(\mathbb{Z}, *)$ is fuzzy reduced with respect to μ. Indeed, for two arbitrary elements x and y in $\mathbb{Z}$, we have $x \sim_{fo} y$ if and only if $\mu(x * a) = \mu(y * a)$, for any $a \in \mathbb{Z}$, where $\mu(x * a) = \{\frac{1}{|x+a|}, \frac{1}{|x+a+1|}, \frac{1}{|x+a+2|}, \ldots\}$ and similarly, $\mu(y * a) = \{\frac{1}{|y+a|}, \frac{1}{|y+a+1|}, \ldots\}$. Since a is an arbitrary integer, for any x and y we always find a suitable integer a such that $x + a > 0$ and $y + a > 0$. This means that the sets $\mu(x * a)$ and $\mu(y * a)$ contain descending sequences of positive integers, so they are equal only when $x = y$. Therefore $x \sim_{fo} y$.

In the following, we will study the fuzzy reducibility of some particular types of finite hypergroups, with respect to the *grade* fuzzy set $\widetilde{\mu}$, defined by Corsini [16]. We recall here its definition. With any crisp hypergroupoid $(H, \circ)$ (not necessarily a hypergroup) we may associate the fuzzy set $\widetilde{\mu}$ considering, for any $u \in H$,

$$\widetilde{\mu}(u) = \frac{\displaystyle\sum_{(x,y)\in Q(u)} \frac{1}{|x \circ y|}}{q(u)}, \tag{3.2}$$

where $Q(u) = \{(a,b) \in H^2 \mid u \in a \circ b\}$ and $q(u) = |Q(u)|$. By convention, we take $\widetilde{\mu}(u) = 0$ anytime when $Q(u) = \emptyset$. In other words, as it is written in [29], the value $\widetilde{\mu}(u)$ represents the average value of reciprocals of the sizes of all hyperproducts $x \circ y$ containing the element u in H. In addition, sometimes when we will refer to formula (3.2), we will denote its numerator by $A(u)$, while the denominator is already denoted by $q(u)$.

Remark 3.3. As already explained in Remark 3.2 (ii), generally, for an arbitrary fuzzy set, $x \sim_i y \not\Rightarrow x \sim_{fi} y$, while the implication holds if we consider the grade fuzzy set $\widetilde{\mu}$. Indeed, if $x \sim_i y$, then $x \in a \circ b$ if and only if $y \in a \circ b$ and therefore $Q(x) = Q(y)$, implying that $q(x) = q(y)$, and moreover $A(x) = A(y)$. This leads to the equality

$\widetilde{\mu}(x) = \widetilde{\mu}(y)$. By consequence, based on Remark 3.2 (iii), it holds $x \sim_{fi} y$, with respect to $\widetilde{\mu}$.

Example 3.3. Let us consider now a total finite hypergroup H, i.e. $x \circ y = H$, for all $x, y \in H$. It is easy to see that $x \sim_e y$ for any $x, y \in H$, meaning that $\acute{x}_e = H$, for any $x \in H$. Thus, a total hypergroup is not reduced. What can we say about the fuzzy reducibility with respect to the grade fuzzy set $\widetilde{\mu}$?

For any $u \in H$, there is

$$\widetilde{\mu}(u) = \frac{|H|^2 \frac{1}{|H|}}{|H|^2} = \frac{1}{|H|}.$$

Since, $x \sim_\circ y$ for any $x, y \in H$, it follows that $x \sim_{fo} y$, for any $x, y \in H$. Then, it is clear that $\widetilde{\mu}(x) = \widetilde{\mu}(y)$, for all $x, y \in H$, implying that $x \sim_{fi} y$, for all $x, y \in H$. Concluding, it follows that any total finite hypergroup is neither reduced, nor fuzzy reduced.

Based now on Remarks 3.1 and 3.3, the following assertion is clear.

Corollary 3.1. *If $(H, \circ)$ is a not reduced hypergoup, then it is also not fuzzy reduced with respect to the grade fuzzy set $\widetilde{\mu}$.*

Before we start to examine reducibility for particular types of hypergroups, we will present an easy result related to the fuzzy reducibility of a subhypergroup of hypergroups.

Proposition 3.1. *A subhypergroup $(K, \circ)$ of a fuzzy reduced hypergroup $(H, \circ)$ is a fuzzy reduced hypergroup.*

Proof. Let the element a be an arbitrary element that belongs to the set K, where $K \subseteq H$. Then the element a also belongs to H, and since $(H, \circ)$ is fuzzy reduced there is $\hat{a}_{fe} = \{a\}$ i.e., $(K, \circ)$ is a fuzzy reduced hypergroup, too. $\square$

Proposition 3.2. *Let $(H, \circ)$ be a proper complete hypergroup and consider on H the grade fuzzy set $\widetilde{\mu}$. Then $\sim \subset \sim_{fe}$ (with respect to the fuzzy set $\widetilde{\mu}$).*

Proof. By the definition of the grade fuzzy set $\widetilde{\mu}$, one obtains that

$$\widetilde{\mu}(x) = \frac{1}{|A_{g_x}|}, \quad \text{for any} \quad x \in H. \tag{3.3}$$

Take now $x, y \in H$ such that $x \sim y$. There exists $g_x \in G$ such that $x, y \in A_{g_x}$, thereby $\widetilde{\mu}(x) = \widetilde{\mu}(y)$ and by Remark 3.2 (iii) we have $x \sim_{fi} y$. Moreover, by Proposition 2.17

there is $x \sim_o y$ and by Remark 3.1 we get that $x \sim_{fo} y$. Concluding, we have proved that $x \sim y \Rightarrow x \sim_{fe} y$, with respect to $\widetilde{\mu}$. □

Theorem 3.1. *Any proper complete hypergroup is not fuzzy reduced with respect to the grade fuzzy set* $\widetilde{\mu}$.

Proof. Since $(H, \circ)$ is not reduced hypergroup, there exist elements $a \neq b \in H$, where $a, b \in H$ such that $a \sim b$. By the Proposition 3.6, $a \sim_e b$ implies $a \sim_{fe} b$, meaning that $(H, \circ)$ is not fuzzy reduced with respect to the grade fuzzy set $\widetilde{\mu}$. □

3.1 Fuzzy reducibility in i.p.s. hypergroups

In the following we will discuss the fuzzy reducibility of i.p.s. hypergroups with respect to the grade fuzzy set $\tilde{\mu}$. In the previous chapter we proved that those hypergroups are reduced.

Theorem 3.2. *Any i.p.s. hypergroup is not fuzzy reduced with respect to the fuzzy set* $\tilde{\mu}$.

Proof. Since any i.p.s. hypergroup contains at least one non-zero scalar, take arbitrarly such a $u \in Sc(H)$. We will prove that $u \sim_{fi} 0$ and $u \sim_{fo} 0$, therefore $|\hat{0}_{fe}| \geq 2$, meaning that H is not fuzzy reduced.

First we will prove that, for any $u \in Sc(H)$, there is $\widetilde{\mu}(0) = \widetilde{\mu}(u)$, equivalently with $u \sim_{fi} 0$. For doing this, based on the fact that $\widetilde{\mu}(x) = \frac{A(x)}{q(x)}$, for all $x \in H$, we show that $A(0) = A(u)$ and $q(0) = q(u)$.

Let us start with the computation of $q(0)$ and $q(u)$. If $0 \in x \circ y$, it follows that $y \in x^{-1} \circ 0$, that is $y = x^{-1}$ and then $Q(0) = \{(x, y) \in H^2 \mid 0 \in x \circ y\} = \{(x, x^{-1}) \mid x \in H\}$. Thereby $q(0) = |Q(0)| = n = |H|$. On the other hand, by Proposition 2.14, we have $q(u) = n = |H|$ (since for any $x \in H$ there exists a unique element $y \in H$ such that u belongs to the set $x \circ y$ [22]).

Let us calculate now $A(0)$. By formula (3.2), we get that

$$A(0) = \sum_{(x,y) \in Q(0)} \frac{1}{|x \circ y|} = \sum_{x \in H} \frac{1}{|x \circ x^{-1}|} =$$

$$\sum_{a \in Sc(H)} \frac{1}{|a \circ a^{-1}|} + \sum_{x \notin Sc(H)} \frac{1}{|x \circ x^{-1}|} = |Sc(H)| + \sum_{x \notin Sc(H)} \frac{1}{|x \circ x^{-1}|}.$$

Since, for $u \in Sc(H), \exists x \notin Sc(H)$ such that $u \in x \circ x^{-1} \cap I_{ps}(x)$, we similarly get that

$$A(u) = |Sc(H)| + \sum_{x \notin Sc(H)} \frac{1}{|x \circ x^{-1}|}$$

and it is clear that $A(0) = A(u)$, so $\widetilde{\mu}(0) = \widetilde{\mu}(u)$. Therefore $0 \sim_{fi} u$.

It remains to prove the second part of the theorem, that is $0 \sim_{fo} u$, equivalently with $\widetilde{\mu}(0 \circ x) = \widetilde{\mu}(u \circ x), \forall x \in H$.

If $x \in Sc(H)$, then $u \circ x \in Sc(H)$ and by the first part of the theorem, there is $\widetilde{\mu}(u \circ x) = \widetilde{\mu}(0) = \widetilde{\mu}(x) = \widetilde{\mu}(0 \circ x)$.

If $x \notin Sc(H)$, then since $Sc(H) \subset I_{ps}(a)$, for any $a \notin Sc(H)$, it follows that $Sc(H) \subset I_{ps}(x)$, so $u \circ x = x$, and then $\widetilde{\mu}(u \circ x) = \widetilde{\mu}(x) = \widetilde{\mu}(0 \circ x)$. Now the proof is complete.

□

Example 3.4. Let the i.p.s hypergroup be given by the following table [25]

H	0	1	2	3	4
0	0	1	2	3	4
1	1	$0,2$	1	3	4
2	2	1	0	3	4
3	3	3	3	4	$0,1,2$
4	4	4	4	$0,1,2$	3

Notice that 0 is the unique scalar identity since $0 \circ x = x$ for all $x \in \{0,1,2,3,4\}$. Besides, the element 2 is a scalar with $|2 \circ x| = |x \circ 2| = 1$. Following the proof of Theorem 2.13, we have that $0 \sim_{fe} 2$. Indeed, $\widetilde{\mu}(0) = \frac{1 \cdot 2 + \frac{1}{2} + \frac{1}{3} \cdot 2}{5} = \widetilde{\mu}(2)$. Thus, $\widetilde{\mu}(0)$ and $\widetilde{\mu}(2)$ appear in the same sets $\widetilde{\mu}(a \circ b)$, where $a, b \in H$, i.e., $0 \sim_{fi} 2$. Also, $\widetilde{\mu}(0 \circ x) = \widetilde{\mu}(2 \circ x)$ for all $x \in H$. This is obvious for the elements $1,3,4$ since $0 \circ x = 2 \circ x$, for $x \in \{1,3,4\}$. Notice that $\widetilde{\mu}(0 \circ 0) = \widetilde{\mu}(2 \circ 0) = \widetilde{\mu}(0) = \widetilde{\mu}(2)$. Similarly, $\widetilde{\mu}(0 \circ 2) = \widetilde{\mu}(2 \circ 2)$. From here, it follows that $0 \sim_{f0} 2$. Hence, $\hat{0}_{fe} = \hat{2}_{fe} = \{0,2\}$ i.e., the hypergroup is not fuzzy reduced.

3.2 Fuzzy reducibility in non-complete 1-hypergroups

In this subsection we study the reducibility and fuzzy reducibility with respect to the grade fuzzy set for some particular finite non-complete 1-hypergroups defined and investigated by Corsini and Cristea [17]. Recall that the hypergroup H is called *1-hypergroup* if the cardinality of its heart ω_H is 1 [13].

Let us describe the procedure of the construction of the above mentioned hypergroup. Consider the set $H = H_n = \{e\} \cup A \cup B$, where $A = \{a_1, \ldots, a_\alpha\}$ and $B = \{b_1, \ldots, b_\beta\}$, with $\alpha, \beta \geq 2$ and $n = \alpha + \beta + 1$, such that $A \cap B = \emptyset$ and $e \notin A \cup B$. Define on H the hyperoperation "$\circ$" by the following rule [17]:

- for all $a \in A$, $a \circ a = b_1$,
- for all $(a_1, a_2) \in A^2$ such that $a_1 \neq a_2$, set $a_1 \circ a_2 = B$,
- for all $(a, b) \in A \times B$, set $a \circ b = b \circ a = e$,
- for all $(b, b') \in B^2$, there is $b \circ b' = A$,
- for all $a \in A$, set $a \circ e = e \circ a = A$,
- for all $b \in B$, $b \circ e = e \circ b = B$ and
- $e \circ e = e$.

H_n is an 1-hypergroup which is not complete.

We will discuss the (fuzzy) reducibility of this hypergroup for different cardinalities of the sets A and B.

1) Suppose now that $n = |H_6| = 6$, where $H = H_6 = e \cup A \cup B$, $\alpha = |A| = 2$, $\beta = |B| = 3$, $A \cap B = \emptyset$, $e \notin A \cup B$ with $A = \{a_1, a_2\}$, $B = \{b_1, b_2, b_3\}$. Thus the Cayley table of $(H_6, \circ)$ is the following one

H	e	a_1	a_2	b_1	b_2	b_3
e	e	A	A	B	B	B
a_1		b_1	B	e	e	e
a_2			b_1	e	e	e
b_1				A	A	A
b_2					A	A
b_3						A

From the above Cayley table, we notice immediately that the elements b_2 and b_3 are essentially indistinguishable, while the equivalence class with respect to the $\sim_e$ of all other elements is a singleton. Thereby, H is not reduced.

Calculating now the values of the grade fuzzy set $\widetilde{\mu}$, one obtains $\widetilde{\mu}(e) = 1, \widetilde{\mu}(a_1) = \widetilde{\mu}(a_2) = 0.5, \widetilde{\mu}(b_1) = 0.467, \widetilde{\mu}(b_2) = \widetilde{\mu}(b_3) = 0.333$. Since $\widetilde{\mu}(a_1) = \widetilde{\mu}(a_2)$, it follows that $a_1 \sim_{fi} a_2$. But $\widetilde{\mu}(a_1 \circ a_1) = \widetilde{\mu}(\{b_1\}) = \{\widetilde{\mu}(b_1)\}$, while $\widetilde{\mu}(a_1 \circ a_2) = \widetilde{\mu}(B) = \{\widetilde{\mu}(b_1), \widetilde{\mu}(b_2)\}$, so $\widetilde{\mu}(a_1 \circ a_1) \neq \widetilde{\mu}(a_1 \circ a_2)$, meaning that $a_1 \not\sim_{fo} a_2$, that is $a_1 \not\sim_{fe} a_2$.

On the other side, we have $b_2 \sim_{fo} b_3$ because they are also operationally equivalent, and $b_2 \sim_{fi} b_3$ because $\widetilde{\mu}(b_2) = \widetilde{\mu}(b_3)$. This is equivalent with $b_2 \sim_{fe} b_3$ and therefore H is not fuzzy reduced with respect to $\widetilde{\mu}$.

2) Consider now the most general case. The Cayley table of the hypergroup H is the following one:

H	e	a_1	a_2	$\cdots$	a_α	b_1	b_2	$\cdots$	b_β
e	e	A	A	$\cdots$	A	B	B	$\cdots$	B
a_1		b_1	B	$\cdots$	B	e	e	$\cdots$	e
a_2			b_1	$\cdots$	B	e	e	$\cdots$	e
$\vdots$				$\ddots$					$\vdots$
a_α					b_1	e	e	$\cdots$	e
b_1						A	A	$\cdots$	A
b_2							A	$\cdots$	A
$\vdots$								$\ddots$	$\vdots$
b_β									A

As already calculated in [17], there is $\widetilde{\mu}(e) = 1$, $\widetilde{\mu}(a_i) = \frac{1}{\alpha}$, for any $i = 1, \ldots, \alpha$, while $\widetilde{\mu}(b_1) = \dfrac{\alpha^2 + \alpha\beta + 2\beta - \alpha}{\beta(\alpha^2 + 2\beta)}$, and $\widetilde{\mu}(b_j) = \frac{1}{\beta}$, for any $j = 2, \ldots, \beta$. As in the previous case, we see that any two elements in $B \setminus \{b_1\}$ are operational equivalent, by consequence also fuzzy operational equivalent, and indistinguishable and fuzzy indistinguishable (because their values under the grade fuzzy set $\widetilde{\mu}$ are the same). Concluding, this non complete 1-hypergroup is always not reduced, either not fuzzy reduced with respect to $\widetilde{\mu}$.

Remark 3.4. The hypergroups defined by the above described method are not the only one non-complete $1-$hypergroups. In the following example we will present a non-complete $1-$hypergroup which is both, reduced and fuzzy reduced with respect to the grade fuzzy set $\widetilde{\mu}$.

Example 3.5. Let the hypergroup $(H, \circ)$ be given by the following table

H	e	a	b	c	d
e	e	a, b	a, b	c, d	c, d
a	a, b	c, d	c	e	e
b	a, b	c	c, d	e	e
c	c, d	e	e	a, b	a, b
d	c, d	e	e	a, b	a, b

Notice that it holds that $c \sim_o d$, but $c \in a \circ b$, while $d \notin a \circ b$, which shows that $c \not\sim_i d$ and consequently, $c \not\sim_e d$. Since the rows corresponding to the elements a, b and c are different, those elements have singleton equivalence classes with respect to the essential indistinguishability. Hence, $(H, \circ)$ is a reduced hypergroup.
By calculating the values of the elements $x \in H$ under the grade fuzzy set $\widetilde{\mu}$, one obtains $\widetilde{\mu}(a) = \widetilde{\mu}(b) = \widetilde{\mu}(d) = \frac{1}{2}, \widetilde{\mu}(e) = 1, \widetilde{\mu}(c) = \frac{5}{8}$. Since the first two rows are the same, then we get that $\widetilde{\mu}(c \circ x) = \widetilde{\mu}(d \circ x)$, for all $x \in H$. However, $\widetilde{\mu}(c) \in \widetilde{\mu}(a \circ b)$, while $\widetilde{\mu}(d)$ does not belong to the same set. Hence, $c \not\sim_{fe} d$. Similarly we prove that for arbitrary two elements $x, y \in H$ there is $x \not\sim_{fo} y$, which finally gives that $x \not\sim_{fe} y$. We conclude that the hypergroup $(H, \circ)$ is fuzzy reduced with respect to the grade fuzzy set $\widetilde{\mu}$.

In the following we will show some results regarding the fuzzy reducibility in cyclic hypergroups. Again we will consider the fuzzy reducibility with respect to the grade fuzzy set $\widetilde{\mu}$.

Theorem 3.3. [1] *There are only five commutative single power cyclic hypergroups of order two up to isomorphism.*

Al-Tahan and Davvaz [1] listed all such hypergroups, with $H = \{a, b\}$ as follows

$\circ_1$	a	b
a	H	H
b	H	H

$\circ_2$	a	b
a	H	H
b	H	a

$\circ_3$	a	b
a	H	b
b	b	a

$\circ_4$	a	b
a	H	H
b	H	b

$\circ_5$	a	b
a	H	a
b	a	b

(3.4)

Proposition 3.3. *All commutative single power cyclic hypergroups of order 2 except the total hypergroup are fuzzy reduced with respect to the grade fuzzy set $\widetilde{\mu}$.*

Proof. Since Davvaz and Al-Tahan listed all commutative single power cyclic hypergroups of order two, easy calculations of $\widetilde{\mu}(x)$ for all $x \in H$ prove this proposition. $\square$

Notice that, except the total hypergroup, all other hypergroups listed above are reduced hypergroups. Let us show the fuzzy reducibility of a particular single power cyclic hypergroup of order 2.

Example 3.6. Let the hypergroup $(H, \circ)$ be given with the following table

$\circ$	a	b
a	H	H
b	H	b

(3.5)

Here, $\widetilde{\mu}(a) = \frac{1}{2}, \widetilde{\mu}(b) = \frac{5}{8}$. Since $\widetilde{\mu}(a \circ b) = \{\frac{1}{2}, \frac{5}{8}\} \neq \widetilde{\mu}(b \circ b) = \frac{5}{8}$, then $a \nsim_{fo} b$, which immediately gives that $\hat{a}_{fe} = \{a\}, \hat{b}_{fe} = \{b\}$.

In the next example we will present a fuzzy reduced cyclic hypergroup which is not s-cyclic, nor single power cyclic.

Example 3.7. Let $(H, \circ)$ be the following hypergroup

$\circ$	1	2	3	4
1	1	1	1, 2, 3	1, 2, 4
2	1	1	1, 2, 3	1, 2, 4
3	1, 2, 3	1, 2, 3	1, 2, 3	3, 4
4	1, 2, 3	1, 2, 4	3, 4	1, 2, 4

(3.6)

Using easy calculations, we obtain that $\widetilde{\mu}(3) = \widetilde{\mu}(4) = \frac{8}{21}$, which shows that $3 \sim_{fi} 4$. However, $\widetilde{\mu}(3 \circ 3) = \{\frac{8}{21}, \frac{11}{21}\} \neq \widetilde{\mu}(4 \circ 3) = \{\frac{8}{21}\}$. Thus, $3 \nsim_{fi} 4$, implying that $3 \nsim_{fe} 4$, thus the hypergroup $(H, \circ)$ is fuzzy reduced with respect to the grade fuzzy set $\widetilde{\mu}$.

Let us present an example of a non-fuzzy reduced hypergroup which is a join space.

Example 3.8. Let ρ be a reflexive and symmetric relation given on the set $H = \{x, y, z\}$ with: $\rho = \{(x, x), (y, y), (z, z), (x, z), (z, x), (y, z), (z, y)\}$.
Let us consider the hyperoperation $\circ$ given on the H with:

$$\forall (x, y) \in H^2, x \circ x = L_x, x \circ y = L_x \cup L_y.$$

In the previous chapter we proved that the hypergroup $(H_\rho, \circ)$ is a reduced hypergroup. Let us form the Cayley table of a hypergroup for an easier computation of the grade fuzzy set for any $x \in H$.

$\circ$	x	y	z
x	x, z	H	H
y	H	y, z	H
z	H	H	H

(3.7)

Calculations give that $\widetilde{\mu}(x) = \frac{17}{48} = \widetilde{\mu}(y)$, which implies that $x \sim_{fe} y$. Besides, $\widetilde{\mu}(x \circ a) = \widetilde{\mu}(y \circ a)$ and $\widetilde{\mu}(a \circ x) = \widetilde{\mu}(a \circ y)$, for any $a \in \{x, y, z\}$. Hence, the hypergroup $(H, \circ)$ is not-fuzzy reduced.

3.3 Fuzzy reducibility in Corsini hypergroups

The intent of this section is proving that a Corsini hypergroup $(H, \circ)$ is not fuzzy reduced with respect to the grade fuzzy set $\tilde{\mu}$. For doing this, first we present some properties regarding the hyperproducts $x_i \circ x_i$, with $x_i \in H$. For a finite hypergroup H with n elements, we will denote its cardinality by $|H| = n$.

Definition 3.3. [29] *Let $\Gamma = (H; \{A_i\}_i)$ be a hypergraph, i.e., for any i, $A_i \in \mathcal{P}(H) \setminus \emptyset; \bigcup_i A(i) = H$ for any $x \in H$. Set $E(x) = \bigcup_{x \in A_i} A_i$. The hypergroupoid $H_\Gamma = (H, \circ)$ where the hyperoperation $\circ$ is defined by:*

$$\forall (x, y) \in H^2, \quad x \circ y = E(x) \cup E(y)$$

is called a hypergraph hypergroupoid.

Definition 3.4. [15] *The hypergroupoid H_Γ satisfies for each $(x, y) \in H^2$, the following conditions:*

1. $x \circ y = x \circ x \cup y \circ y$,
2. $x \in x \circ x$,
3. $y \in x \circ x$ *if and only if* $x \in y \circ y$.

Theorem 3.4. [15] *A hypergroupoid $(H, \circ)$ satisfying the conditions in Definition 3.4 is a hypergroup if and only if also the following condition is valid:*

$$\forall (a, c) \in H^2 \quad c \circ c \circ c \setminus c \circ c \subseteq a \circ a \circ a.$$

Proposition 3.4. *Let $(H, \circ)$ be a Corsini hypergroup with n elements. If an element x_i appears in exactly k hyperproducts $x_j \circ x_j$, $j = 1, 2, \ldots, n$, then $q(x_i) = 2nk - k^2$.*

Proof. Let x_i be an arbitrary element from $H = \{x_1, x_2, x_3, \ldots, x_n\}$ which appears in k hyperproducts $x_j \circ x_j$, for some $j = 1, \ldots, n$. By the definition of the hyperoperation of a Corsini hypergroup, it follows that x_i appears in every hyperproduct $x_j \circ x_k$, with $k \in \{1, \ldots, n\}$. For one fixed k, because the commutativity, x_i appears in $n + n - 1$ hyperproducts. The sum of all such cases is:

$$(2n-1) + (2n-1) - 2 \cdot 1 + (2n-1) - 2 \cdot 2 + \ldots + (2n-1) - 2 \cdot (k-1) =$$

$$(2n-1) \cdot k - 2(1 + 2 + \ldots + k - 1) = (2n-1) \cdot k - (k-1)k = 2nk - k^2.$$

□

Proposition 3.5. *The sum of all cardinalities of $x_i \circ x_i$ with $x_i \in H$ when $|H|$ is odd (even) is an odd (even) number.*

Proof. Let $|H| = n$ be an even number. If $|x_i \circ x_i| = 1$, for every $x_i \in H$, then $\sum_i^n |x_i \circ x_i| = 1 \cdot n = n$ which is an even number. Let add k elements to a hyperproduct $x_i \circ x_i, k \leq n - 1$. In that case, by the property 3 of the definition of the hyperoperation "$\circ$", we have to add the element x_i to $k-$ hyperproducts $x_j \circ x_j$. All together, we add $k + k = 2k$ elements, which is again an even number. Continuing this process, so adding an arbitrary number of elements to any hyperproduct $x_i \circ x_i$, we always get an even number. Summing arbitrary even numbers, we obtain at the end an even number. The proof is analogous in the case when n is an odd number. □

Proposition 3.6. *Let $(H, \circ)$ be a Corsini hypergroupoid of cardinality n. The number of all possible different sums of the cardinalities of the hyperproducts $x_i \circ x_i, x_i \in H$, is $\frac{n^2-n}{2} + 1$.*

Proof. The proof will be performed using the mathematical induction. For hypergroups of cardinality 2, the property is easily satisfied, because if H contains two elements, we have exactly two possibilities. The hyperproducts $x \circ x$ are singleton, or equal to H. In the first case the sum of the cardinalities of $x_i \circ x_i$ is 2, while in the second case the sum is 4. Thus, the number of the different sums is 2, i.e. $\frac{2^2-2}{2} + 1$. Assume that for $|H| = n$ the number of the different sums is equal to $\frac{n^2-n}{2} + 1$. Let us prove that the claim is valid for $|H| = n + 1$. In this case we have to analyse only the hyperproduct $x_{n+1} \circ x_{n+1}$. If $x_{n+1} \circ x_{n+1} = \{x_{n+1}\}$, then we have $\frac{n^2-n}{2} + 1$ possible sums, i.e. the number of sums is the same as in the inductive case. The other cases are: $x_{n+1} \circ x_{n+1} = \{x_{n+1}, x_i\}, x_{n+1} \circ x_{n+1} = \{x_{n+1}, x_i, x_j\}, \ldots, x_{n+1} \circ x_{n+1} = H$. It gives n sums more, which is finally $\frac{n^2-n}{2} + 1 + n$, i.e. number of possible sums when $|H| = n+1$ is equal to $\frac{(n+1)^2-(n+1)}{2} + 1$, which proves the proposition.

□

Remark 3.5. Let $(H, \circ)$ be a Corsini hypergroup of cardinality n. There are at least $\frac{n^2-n}{2}+1$ Corsini hypergroups of order n up to isomorphism. Since the hyperproducts $x \circ x, x \in H$ completely determine the hypergroup, it folows that $\frac{n^2-n}{2}+1$ different sums define at least as many different hypergroups. One sum can form more different tables, and in case when $n \geq 3$ the number of hypergroups is greater.

Proposition 3.7. *Let $(H, \circ)$ be a Corsini hypergroup of cardinality n. If an element x_i appears in k hyperproducts $x_j \circ x_j$, and if we assume that the cardinalities of those sets are, respectively $m_1, m_2, \ldots, m_k$, then*

$$\widetilde{\mu}(x_i) = \frac{\frac{1}{m_1} + \frac{1}{m_2} + \ldots + \frac{1}{m_k} + 2 \cdot \sum\limits_{\substack{i \neq j \\ i=1,\ldots,k}} \frac{1}{|x_i \circ x_j|}}{2nk - k^2}$$

Proof. According to definition of the fuzzy grade set $\widetilde{\mu}$ and Proposition 3.4, the result is clearly satisfied. □

Remark 3.6. If two elements of a Corsini hypergroup have the same number of appearances in some hyperproducts $x_j \circ x_j$, and the cardinalities of those hyperproducts are the same for both elements, based on Proposition 3.7, then their values under the grade fuzzy set $\widetilde{\mu}$ are the same. Hereinafter, we will say that elements with this property are in the *same* formation.

Proposition 3.8. *In any Corsini hypergroup $(H, \circ)$, the fuzzy operational equivalence implies the fuzzy inseparability.*

Proof. Let $x, y \in H$ be two arbitrary elements in H such that $x \sim_{fo} y$, i.e., $\widetilde{\mu}(x \circ a) = \widetilde{\mu}(y \circ a)$, for $\forall a \in H$. It means that:

$$\widetilde{\mu}(x \circ x \cup a \circ a) = \widetilde{\mu}(y \circ y \cup a \circ a)$$

$\Longleftrightarrow$

$$\widetilde{\mu}(x \circ x) \cup \widetilde{\mu}(a \circ a) = \widetilde{\mu}(y \circ y) \cup \widetilde{\mu}(a \circ a).$$

Since this equality is satisfied for every set $\widetilde{\mu}(a \circ a), a \in H$, it follows that $\widetilde{\mu}(x \circ x) = \widetilde{\mu}(y \circ y)$ and contains both $\widetilde{\mu}(x)$ and $\widetilde{\mu}(y)$ by property 3. of Definition 3.4. If $\widetilde{\mu}(x) = \widetilde{\mu}(y)$, then clearly $x \sim_{fi} y$. Let us consider now the case when $\widetilde{\mu}(x) \neq \widetilde{\mu}(y)$. Suppose that $\widetilde{\mu}(x) \in \widetilde{\mu}(c \circ d) = \widetilde{\mu}(c \circ c) \cup \widetilde{\mu}(d \circ d)$. Let us take $\widetilde{\mu}(x) \in \widetilde{\mu}(c \circ c)$. It follows that, for some $z \in c \circ c, \widetilde{\mu}(x) = \widetilde{\mu}(z)$. The equality $\widetilde{\mu}(x \circ x) = \widetilde{\mu}(y \circ y)$ means that $\{\widetilde{\mu}(l) \mid l \in x \circ x\} = \{\widetilde{\mu}(k) \mid k \in y \circ y\}$, i.e., for every $l \in x \circ x$ there exists $k \in y \circ y$ such that $\widetilde{\mu}(l) = \widetilde{\mu}(k)$. Now, since $\widetilde{\mu}(x) = \widetilde{\mu}(z) \in \widetilde{\mu}(x \circ x)$, $\widetilde{\mu}(x) \in \widetilde{\mu}(x \circ x)$, and $\widetilde{\mu}(x \circ x) = \widetilde{\mu}(y \circ y)$, we conclude that $\widetilde{\mu}(z) \in \widetilde{\mu}(y \circ y)$. Thus there exists $l \in y \circ y$ such that $\widetilde{\mu}(z) = \widetilde{\mu}(l)$. But $\widetilde{\mu}(z) \in \widetilde{\mu}(c \circ c)$, so $\widetilde{\mu}(l) \in \widetilde{\mu}(c \circ c)$, with $l \in y \circ y$, which finally gives $\widetilde{\mu}(y) \in \widetilde{\mu}(c \circ c)$. The converse implication can be proved taking $\widetilde{\mu}(y) \in \widetilde{\mu}(c \circ c)$ and proving that $\widetilde{\mu}(x) \in \widetilde{\mu}(c \circ c)$. This shows that $\widetilde{\mu}(x)$ and $\widetilde{\mu}(y)$ appear in the same $\widetilde{\mu}(c \circ c)$. Finally, according to the definition of $\widetilde{\mu}(x \circ y)$, it is easy to prove that the previous equivalence implies the fuzzy inseparability. □

Proposition 3.9. *Let $(H, \circ)$ be a Corsini hypergroup of cardinality n. If x is an element such that $x \circ x$ is a singleton, i.e. $x \circ x = \{x\}$, then* $\widetilde{\mu}(x) = \dfrac{1+2 \cdot \sum\limits_{a \neq x} \dfrac{1}{|x \circ a|}}{2n-1}$, *with $a \in H$.*

Proof. Using Proposition 3.4 we easily get that $q(x) = 2n - 1$. Since x appears in every product $x \circ a, a \in H$, and the commutativity holds, then $A(x) = 1 + 2 \cdot \sum\limits_{a \neq x} \dfrac{1}{|x \circ a|}$, which clearly gives the formula. □

Using this result, we can state sufficient conditions such that two elements in a Corsini hypergroup are fuzzy essentially indistinguishable.

Proposition 3.10. *If there exist two elements x, y in a Corsini hypergroup $(H, \circ)$ such that $x \circ x = x$ and $y \circ y = y$, then $x \sim_{fe} y$.*

Proof. Using Proposition 3.9 this obviously holds, because $\widetilde{\mu}(x) = \widetilde{\mu}(y)$. □

Proposition 3.11. *If there exist two elements x, y in Corsini hypergroup $(H, \circ)$ such that $x \circ x = y \circ y = H$, then $x \sim_{fe} y$.*

Proof. Since $x \circ x = H$, based on condition 3 of Definition 3.4 it follows that x appears in all hyperproducts $z \circ z$, with $z \in H$, and similarly holds for y. So x and y are in the same formation. According to Proposition 3.7, we have $\widetilde{\mu}(x) = \widetilde{\mu}(y)$, so x and y are fuzzy inseparable. Besides, $\mu(x \circ a) = \mu(y \circ a) = \mu(\{x \mid x \in H\})$, which implies the fuzzy operational equivalence. Therefore, $x \sim_{fe} y$. □

Theorem 3.5. *Any B-hypergroup is not fuzzy reduced with respect to the grade fuzzy set $\widetilde{\mu}$.*

Proof. Regarding to the definition of a B-hypergroup, we have $|x \circ x| = 1$ and $|x \circ a| = 2$ for every $x \neq a$, so $A(x) = 1 + 2 \cdot (n-1) \cdot \frac{1}{2} = n$. Using Proposition 3.4, we know that $q(x) = 2n - 1$, which clearly gives that, for any $x \in H, \widetilde{\mu}(x) = \frac{n}{2n-1}$. Hence, two arbitrary elements in a B-hypergroup are fuzzy inseparable. Besides, $\widetilde{\mu}(x \circ a) = \widetilde{\mu}(y \circ a)$, for any $a \in H$ since $\widetilde{\mu}(x) = \widetilde{\mu}(y)$ for two arbitrary elements from H, and $\widetilde{\mu}(x \circ a) = \widetilde{\mu}(\{x, a\}) = \{\widetilde{\mu}(x), \widetilde{\mu}(a)\}$. □

Proposition 3.12. *Let $(H, \circ)$ be a Corsini hypergroup with $|H| \geq 2$. There always exist two elements $x, y \in H$ such that $\widetilde{\mu}(x \circ x) = \widetilde{\mu}(y \circ y)$.*

Proof. We will split the proof in some cases. Using Propositions 3.10 and 3.11 we can eliminate the cases when there exist $x, y \in H$ such that $x \circ x$ and $y \circ y$ are singleton or equal to H. It remains then to consider other three cases.

1. There exists $x \in H$ such that $x \circ x = H$.

2. There exists $x \in H$ such that $x \circ x = x$,

3. The hypergroup doesn't contain any element x such that $x \circ x$ is equal to x or H.

Case 1. Without losing the generality, assume that $H = \{x_1, x_2, \ldots, x_n\}$ and $x_n \circ x_n = H$. This means that any $x_i \in H$ belongs to $x_n \circ x_n$, that implies $x_n \in x_i \circ x_i$, for any $i = 1, 2, \ldots, n$.
Subcase 1.1. If $x_i \circ x_i = \{x_i, x_n\}, i = 1, 2, \ldots, n-1$ and $x_n \circ x_n = H$, then by Proposition 3.7, we know that $\widetilde{\mu}(x_i)$ is the same, for all $i = 1, 2, \ldots, n-1$. This also implies that $\widetilde{\mu}(x_1 \circ x_1) = \widetilde{\mu}(x_2 \circ x_2) = \ldots = \widetilde{\mu}(x_{n-1} \circ x_{n-1})$, which concludes the result.
Subcase 1.2. Extending the previous subcase, that can be considered as a "*base case*", we can analyze now the situation when we add another element x_k, $k \neq n \neq i$, to the hyperproduct $x_i \circ x_i$. This leads to have $x_k \circ x_k = x_i \circ x_i = \{x_k, x_i, x_n\}$, which clearly gives $\widetilde{\mu}(x_i \circ x_i) = \widetilde{\mu}(x_k \circ x_k)$, which proves the proposition. Continuing the process, we can extend now this subcase into two ways:

- by adding another element to a hyperproduct $x \circ x$, with $x \in H \setminus \{x_i, x_k, x_n\}$ and again we obtain the conclusion of the result, or

- by adding a different element x_l to one of the hyperproducts $x_i \circ x_i$ or $x_k \circ x_k$. Suppose that we add it to $x_i \circ x_i$. Thus we get $x_i \circ x_i = \{x_i, x_k, x_l, x_n\}, x_l \circ x_l = \{x_l, x_i, x_n\}, x_k \circ x_k = \{x_k, x_i, x_n\}$, meaning that x_l and x_k are in the same formations, so $\widetilde{\mu}(x_k) = \widetilde{\mu}(x_l)$ and thereby $\widetilde{\mu}(x_k \circ x_k) = \widetilde{\mu}(x_l \circ x_l)$.

Continuing this process by the above described procedure, we will always get two distinct elements such that $\widetilde{\mu}(x \circ x) = \widetilde{\mu}(y \circ y)$. The process is finite, since we stop when we get two hyperproducts $x \circ x = H$.

Case 2. There exists $x_i \in H$ such that $x_i \circ x_i = x_i$. First, the "*base case*" is when all the other hyperproducts $x \circ x$, with $x \in H \setminus \{x_i\}$, contain two elements. This is possible only if the cardinality of H is odd. If the cardinality of H is an even number, the "*base case*" is when one hyperproduct $x_j \circ x_j$, with $j \neq i$, has three elements, and all the other hyperproducts $x \circ x$ have exactly two elements. The value $\widetilde{\mu}(x_i)$ of all elements x_i such that $|x_i \circ x_i| = 2$ is the same. Repeating the same procedure as in Case 1, we will always obtain two elements x and y which satisfy the result.

Case 3. There doesn't exist x_i such that $x_i \circ x_i = H$ nor $x_i \circ x_i = x_i$. The "*base* cases" are exactly the same as in the second case and they depend on the parity of the cardinality of H. For example, in the case when cardinality is an even number, we can set hyperproducts as: $x_1 \circ x_1 = x_2 \circ x_2 = \{x_1, x_2\}$, $x_3 \circ x_3 = x_4 \circ x_4 = \{x_3, x_4\}, \ldots, x_{n-1} \circ x_{n-1} = x_n \circ x_n = \{x_{n-1}, x_n\}$. The values of all $\widetilde{\mu}(x_i)$ are the same for all $i \in \{1, 2, \ldots, n\}$, so $\widetilde{\mu}(x_i \circ x_i)$ are also the same for $i \in \{1, 2, \ldots, n\}$. In the case when the cardinality is an odd number, we can form hyperproducts $x_i \circ x_i$ as in the previous case for $i = 2, \ldots, n-1$, but take $x_1 \circ x_1 = \{x_1, x_2, x_n\}, x_n \circ x_n = \{x_n, x_1\}$. This case reduces to the first case, too. Using already mentioned procedure of constructing other Corsini hypergroups, we will always get two elements x, y such that $\widetilde{\mu}(x \circ x) = \widetilde{\mu}(y \circ y)$.

□

Let us notice that the procedure described above allows us to construct all finite Corsini hypergroups.

Theorem 3.6. *Any Corsini hypergroup is not fuzzy reduced with respect to the grade fuzzy set* $\widetilde{\mu}$.

Proof. According to Proposition 3.12 we can always find two elements x and y such that $\widetilde{\mu}(x \circ x) = \widetilde{\mu}(y \circ y)$. This implies the fuzzy operational equivalence of these two elements. From here, according to Proposition 3.8, we conclude that they are also fuzzy inseparable. Hence, in any Corsini hypergroup there always exist two elements in the same equivalence class with respect to the fuzzy essential indistinguishability, which gives that the hypergroup is not fuzzy reduced, with respect to the grade fuzzy set $\widetilde{\mu}$. □

Remark 3.7. Do to a manner of construction of Corsini hypergroups, showed in the Proposition 3.12, it is easy to conclude that the infinite Corsini hypergroup is also not fuzzy reduced with respect to the $\widetilde{\mu}$.

In the next example we will show non-fuzzy reducibility for the particular Corsini hypergroup.

Example 3.9. Let $(H, \circ)$ is given by the following table

$\circ$	1	2	3	4	5
1	H	H	H	H	H
2	H	$1,2$	$1,2,3$	$1,2,4$	$1,2,5$
3	H	$1,2,3$	$1,3$	$1,3,4$	$1,3,5$
4	H	$1,2,4$	$1,3,4$	$1,4$	$1,4,5$
5	H	$1,2,5$	$1,3,5$	$1,4,5$	$1,5$

(3.8)

Since the elements 2 and 3 are in the same formations, then $\widetilde{\mu}(2) = \widetilde{\mu}(3)$. Precizely, $\widetilde{\mu}(2) = \frac{\frac{1}{2}+6\cdot\frac{1}{3}+9\cdot\frac{1}{5}}{16}$. The values of 2 and 3 under the grade fuzzy set are equal, which implies that $2 \sim_{fi} 3$. Also, $\widetilde{\mu}(2 \circ x) = \widetilde{\mu}(3 \circ x)$, with $x \in H$ which is easy to conclude since the values $\widetilde{\mu}(2)$ and $\widetilde{\mu}(3)$ are equal. Hence, $2 \sim_{fo} 3$. Finally, $2_{fe} = 3_{fe} = \{2, 3\}$, i.e., H is not fuzzy reduced.

Example 3.10. On the set $H = \{1, 2, 3, \ldots, n\}$ let define the hyperoperation $\circ_\rho$ by $x \circ_\rho y = x \circ_\rho x \cup y \circ_\rho y$, where $x \circ_\rho x = \{z \mid x\rho z\}$ and the relation ρ is defined as $x\rho y \iff x \leq y$[59]. Then $(H, \circ_\rho)$ is fuzzy reduced with respect to the grade fuzzy set $\widetilde{\mu}$.

Indeed, note that $i \circ n = \{1, 2, 3, \ldots, max\{i, n\}\}$. Since 1 is the smallest element in the set H, then $1 \circ i = i \circ 1 = \{1, 2, \ldots, i\}$, for any $i \in H$. Here, 1 appears in any hyperproduct, so $q(1) = n^2$, and the cardinalities of the sets where 1 appears are: $1, 2, \ldots, n$, respectively. Similarly, $2 \circ i = i \circ 2 = \{1, 2, 3, \ldots, i\}$, and $q(2) = n^2 - 1$, because 2 doesn't appear only in the hyperproduct $1 \circ 1$. The element 2 appears in the sets of cardinalities $2, 3, 4, \ldots, n-1$ respectively. For an arbitrary element k, we can conclude that it doesn't appear in hyperproducts $j \circ i$ and $i \circ j$ where $i, j \leq k$. Cardinalities of the sets where k appears are $k, k+1, \ldots n$, because k appear in every $i \circ j$, where i or j are greater than or equal to k. The set of cardinality n where k appears is every set $i \circ n$, for any $i \leq n$. Using the commutativity we conclude that we have a $2n-1$ such sets. Similarly, the set of cardinality $n-1$ where k appears is every set $i \circ (n-1), i \leq n-1$ and the number of them is $2(n-1)+1$. Continuing the procedure, we get that the set of cardinality k where k appears is $i \circ k, i \leq k$ and the number of them is $(2k-1)$. Calculating $A(k)$, we get that k appears in $(2k-1)+(2(k+1)-1)+\ldots+(2n-1)$ hyperproducts, which finally gives:

$$\widetilde{\mu}(k) = \frac{\frac{1}{k}\cdot(k+k-1)+\frac{1}{k+1}(k+1+k-1-1)+\ldots+\frac{1}{n}(n+n-1)}{(2k-1)+(2k+1)+(2k+3)+\ldots+(2n-1)}.$$

By summing and arranging members we get $\widetilde{\mu}(k) = \frac{2(n-k+1)-(\frac{1}{k}+\frac{1}{k+1}+\ldots+\frac{1}{n})}{(n+k-1)(n-k+1)}$. By simple calculations it can be proved that $\widetilde{\mu}(k+1) \leq \widetilde{\mu}(k)$, hence k and $k+1$ are not fuzzy essentially indistinguishable. From the previous inequality we have $\widetilde{\mu}(1) \geq \widetilde{\mu}(2) \geq \ldots \geq \widetilde{\mu}(n)$ so the equivalence class of any element in H is a singleton. Hence, $(H, \circ)$ is fuzzy reduced with respect to the grade fuzzy set $\widetilde{\mu}$.

Remark 3.8. Notice that the previous hypergroup is not a Corsini one, but it satisfies the first two conditions of Definition 3.4.

Proposition 3.13. [20] *If $\widetilde{\mu}_1$ and $\widetilde{\mu}_2$ are the grade fuzzy sets of H_1 and H_2, and $\widetilde{\mu}$ is the grade fuzzy set of the direct product $H_1 \times H_2$ then $\widetilde{\mu}(x, y) = \widetilde{\mu}_1(x) \cdot \widetilde{\mu}_2(y), x, y \in H$.*

Proposition 3.14. *Let $(H, \circ_1)$ and $(H, \circ_2)$ be non-fuzzy reduced hypergroups constructed on the support set H with at least two elements. Then the direct product $(H \times H, \circ_1 \times \circ_2)$ is a non-fuzzy reduced hypergroup with respect to the grade fuzzy set $\widetilde{\mu}$.*

Proof. For two elements a and b, we know that $\mu(a \circ b) = \{\mu(x) \mid x \in a \circ b\}$. Since $(H, \circ_1)$ is not fuzzy reduced, assume that x_1, x_2 are two elements such that $x_1 \sim_{fe} x_2$, i.e. $\widetilde{\mu}_1(x_1 \circ_1 a) = \widetilde{\mu}_1(x_2 \circ_1 a)$, for all $a \in H$. Also, $\widetilde{\mu}_1(x_1)$ and $\widetilde{\mu}_1(x_2)$ appear in the same $\widetilde{\mu}_1(a \circ b), a, b \in H$. Similarly, since $(H, \circ_2)$ is not fuzzy reduced, let y_1 and y_2 be elements in H such that they are fuzzy essential indistinguishable. Our goal is to prove that the ordered pairs (x_1, y_1) and (x_2, y_2) are fuzzy essential indistinguishable. Since $(x_1, y_1) \circ_1 \times \circ_2 (a, b) = (x_1 \circ_1 a, y_1 \circ_2 b)$, it follows that $\widetilde{\mu}((x_1, y_1) \circ_1 \times \circ_2 (a, b)) = \{\widetilde{\mu}_1(x) \cdot \widetilde{\mu}_2(y) | x \in x_1 \circ_1 a, y \in y_1 \circ_2 b\}$. Denote the last set with A and the set $\mu((x_2, y_2) \circ_1 \times \circ_2 (a, b))$ with B. Since $x_1 \sim_{fo} x_2$, we have $\{\widetilde{\mu}_1(x) \mid x \in x_1 \circ_1 a\} = \{\widetilde{\mu}_1(y) : y \in x_2 \circ_1 a\}$, and $y_1 \sim_{fo} y_2$ implies $\{\widetilde{\mu}_2(x) \mid x \in y_1 \circ_2 b\} = \{\widetilde{\mu}_2(y) \mid y \in y_2 \circ_2 b\}$, meaning that $A = B$. This proves the fuzzy operational equivalence of the corresponding elements. For the proof of the fuzzy inseparability, let a, c be elements from H such that $\widetilde{\mu}_1(x_1) \in \widetilde{\mu}_1(a \circ_1 c)$. From here, due to the fuzzy inseparability in $(H, \circ_1)$, $\widetilde{\mu}_1(x_2)$ belongs to the same set. On the other side, let b, d be elements from H such that $\widetilde{\mu}_2(y_1) \in \widetilde{\mu}_2(b \circ_2 d)$, from where we conclude that $\widetilde{\mu}_2(y_2) \in \widetilde{\mu}(b \circ_2 d)$. Using the last two implications, we get:

$$\widetilde{\mu}_1(x_1) \cdot \widetilde{\mu}_2(y_1) \in \{\widetilde{\mu}_1(x) \cdot \widetilde{\mu}_2(y) : x \in a \circ_1 c, y \in b \circ_2 d\} =$$

$$\{\widetilde{\mu}(x, y) : x \in a \circ_1 c, y \in b \circ_2 d\} = \widetilde{\mu}(a \circ_1 c, b \circ_2 d)$$

This means that $\widetilde{\mu}(x_1, y_1) \in \widetilde{\mu}(a \circ_1 c, b \circ_2 d)$. The above mentioned implications show that $\widetilde{\mu}(x_2, y_2)$ belongs to the same set. Similarly, one proves the converse implication. Hence, (x_1, y_1) and (x_2, y_2) are fuzzy inseparable and therefore, $(H, \circ_1)$ and $(H, \circ_2)$ are not fuzzy reduced.

□

The converse of Proposition 3.14 doesn't hold, as we can see in Examples 3.11 and 3.12.

Example 3.11. Let $(H, \circ_1)$ and $(H, \circ_2)$ be hypergroups, where the hyperoperations " $\circ_1$ " and " $\circ_2$ " are defined by the following tables.

$\circ_1$	a	b	c	d
a	a	a	a,b,c	a,b,d
b	a	a	a,b,c	a,b,d
c	a,b,c	a,b,c	a,b,c	c,d
d	a,b,d	a,b,d	c,d	a,b,d

$\circ_2$	a	b	c	d
a	b	b	a,b,c	a,b,d
b	b	b	a,b,c	a,b,d
c	a,b,c	a,b,c	a,b,c	c,d
d	a,b,d	a,b,d	c,d	a,b,d

Here, we will consider fuzzy reducibility with respect to the grade fuzzy set $\widetilde{\mu}$.

By easy calculations, we get: $\widetilde{\mu}_1(a) = \frac{11}{21}, \widetilde{\mu}_1(b) = \frac{1}{3}, \widetilde{\mu}_1(c) = \frac{8}{21}, \widetilde{\mu}_1(d) = \frac{8}{21}$. We can notice that the only rows which are the same are those corresponding to a and b. This implies $a \sim_o b$, which easily gives $a \sim_{fo} b$, but here, $\widetilde{\mu}(a)$ belongs to $\widetilde{\mu}(a \circ a)$, while $\widetilde{\mu}(b)$ does not belong to it, so $a \nsim_{fi} b$. Hence, $a \nsim_{fe} b$. It is easy to see that except a and b all other pairs of elements are not fuzzy operational equivalent, which, together with $a \nsim_{fe} b$ implies that $\hat{x}_{fe} = \{x\}$, for all $x \in H$. Hence, $(H, \circ_1)$ is fuzzy reduced.

Regarding $(H, \circ_2)$, due to the isomorphism of hypergroups, we get the same values of the elements under the fuzzy grade $\widetilde{\mu}_2$. At the same way as for the previous hypergroup, we can conclude that $(H, \circ_2)$ is fuzzy reduced.

Here, $(a, a) \sim_{fo} (b, b)$, because $\widetilde{\mu}((a, a)\circ_1 \times \circ_2(m, n)) = \{\widetilde{\mu}_1(x) \cdot \widetilde{\mu}_2(y) \mid x \in a \circ_1 m, y \in a \circ_2 n\} = \{\widetilde{\mu}_1(x) \cdot \widetilde{\mu}_2(y) \mid \widetilde{\mu}_1(x) \in \{\frac{11}{21}, \frac{1}{3}, \frac{8}{21}\}, \widetilde{\mu}_2(y) \in \{\frac{1}{3}, \frac{11}{21}, \frac{8}{21}\}\}$, where $m, n \in \{a, b, c, d\}$. This set is equal to $\widetilde{\mu}((b, b)\circ_1 \times \circ_2(m, n))$.

Further more, $\widetilde{\mu}(a, a) = \widetilde{\mu}_1(a) \cdot \widetilde{\mu}_2(a) = \frac{11}{21} \cdot \frac{1}{3} = \widetilde{\mu}(b, b)$, which ensures that $(a, a) \sim_{f.i} (b, b)$. Hence, we got a non-fuzzy reduced hypergroup as a direct product of two fuzzy reduced hypergroups.

Example 3.12. Let $(H, \circ_1)$ and $(H, \circ_2)$ be hypergroups, where the hyperoperations " $\circ_1$ " and " $\circ_2$ " are defined by the following tables:

$\circ_1$	a	b	c
a	a,b	a,b	H
b	a,b	a,b	H
c	H	H	c

$\circ_2$	a	b	c
a	a	a	H
b	a	a	H
c	H	H	H

Easy calculations of the fuzzy grade sets $\widetilde{\mu}_1$ and $\widetilde{\mu}_2$ show that the first hypergroup $(H, \circ_1)$ is not fuzzy reduced, while $(H, \circ_2)$ is fuzzy reduced with respect to the grade fuzzy set $\widetilde{\mu}$. As in the previous example, it can be shown that $(b, a) \sim_{fe} (a, a)$, which proves the non-fuzzy reducibility of $(H \times H, \circ_1 \times \circ_2)$.

Proposition 3.15. *The direct product of two Corsini hypergroups is non-fuzzy reduced with respect to the grade fuzzy set $\widetilde{\mu}$.*

Proof. Since an arbitrary Corsini hypergroup is not fuzzy reduced according to Theorem 3.6, using Proposition 3.14 it follows that the direct product of two Corsini hypergroups is not fuzzy reduced. □

Corollary 3.2. *The direct product of a Corsini hypergroup and a total hypergroup is non-fuzzy reduced with respect to the grade fuzzy set $\widetilde{\mu}$.*

Proof. This is a direct consequence of Theorem 1.2. □

As we have already mentioned, the fuzzy aspect of reducibility could be investigated in two directions. Until now, we have studied indistinguishability between the images of the elements of a crisp hypergroup through a fuzzy set, i.e., we have studied the fuzzy reducibility in hypergroups. In the following we will briefly explain the second approach in fuzzyfication of the reducibility concept, i.e., we will consider the reducibility in fuzzy hyperstructures. In this case, the indistinguishability is investigated between the elements of a fuzzy hypergroup. Fuzzy hypergroup is a hypergroup endowed with a fuzzy hyperoperation. This approach will be the topic in our further research. In the next section we will give a definition of a reduced fuzzy hypergroup and present some basic examples.

3.4 Reduced fuzzy hypergroups

In order to define a reduced fuzzy hypergroup, we introduce new equivalence relations on fuzzy hypergroup, i.e., on a hypergroup endowed with a fuzzy hyperoperation. The relations have the same names as in the crisp case: operational equivalence, inseparability and essential indistinguishability.

Definition 3.5. [22] *Two elements x, y in a hypergroup $(H, \circ)$ are called:*

1. *operationally equivalent or by short o-equivalent, and write $x \sim_o y$, if $(x \circ a)(r) = (y \circ a)(r)$, and $(a \circ x)(r) = (a \circ y)(r)$, for any $a, r \in H$;*

2. *inseparable or by short i-equivalent, and write* $x \sim_i y$, *if, for all* $a, b \in H, x \in supp(a \circ b) \Longleftrightarrow y \in supp(a \circ b)$, *i.e.* $(a \circ b)(x) \neq 0 \Longleftrightarrow (a \circ b)(y) \neq 0$;

3. *essentially indistinguishable or by short e-equivalent, and write* $x \sim_e y$, *if they are operationally equivalent and inseparable.*

Definition 3.6. [22] $(H, \circ)$ *is a reduced fuzzy hypergroup if and only if for any* $x \in H$ *there is* $\hat{x}_e = \{x\}$.

Chapter 4

Reducibility in hyperrings

This chapter deals with the study of the reducibility in hyperrings. We consider different classes of general hyperrings and study their reducibility. Also, we determine specific relations between equivalence relations in certain hyperrings.

4.1 Reducibility in hyperrings

It is important to stress that in a semigroup (group), the equivalences $\sim_o$ and $\sim_i$ are exactly the same as the equality relation, meaning that, $x \sim_o y \iff x \sim_i y \iff x = y$ and therefore it is not significant to study the reducibility in hyperrings where the referential set is equipped with a hyperoperation and an operation. More precisely, in a Krasner hyperring or in a multiplicative hyperring it is not worth studying reducibility. Thereby, we will study the reducibility only in general hyperrings, where addition and multiplication are both hyperoperations.
Let us extend now the concept of the reducibility to hyperrings. For any element $x \in R$, we denote by $\hat{x}_r^{\oplus}$ and $\hat{x}_r^{\oplus}$, the equivalence classes of x with respect to the hyperoperations $\oplus$ and $\odot$, respectively, where $r \in \{o, i, e\}$ denotes the type of the equivalence that we consider in Definition 4.1. Taking into account that is not worth studying reducibility in hyperrings containing an operation, we emphasize that the following conclusions regarding the reducibility refer to general hyperrings.

Definition 4.1. [21] *We say that two elements x and y in a hyperring $(R, \oplus, \odot)$ are operationally equivalent, inseparable or essential indistinguishable if they have the same property with respect to hyperoperations, i.e.*

1. $x \sim_o y$ *if* $x \oplus a = y \oplus a, a \oplus x = a \oplus y$ *and* $a \odot x = a \odot y, x \odot a = y \odot a$, *for all* $a \in R$.

2. *$x \sim_i y$ if $x \in a \oplus b \iff y \in a \oplus b$ and $x \in c \odot d \iff y \in c \odot d$, for all $a, b, c, d \in R$.*

3. *Moreover, $x \sim_e y$ if $x \sim_o y$ and $x \sim_i y$.*

Similar to the hypergroup, we introduce the definition of a reduced hyperring, using above defined equivalences.

Definition 4.2. [21] *A hyperring R is a reduced hyperring if the equivalence class of each element $x \in R$ with respect to the essentially indistinguishable relation $\sim_e$ is a singleton, i.e., $\hat{x}_e = \{x\}$ for any $x \in R$.*

The equivalence class of any x in R with respect to the essential indistinguishability $\sim_e$ is obtained as $\hat{x}_e = \hat{x}_e^{\oplus} \cap \hat{x}_e^{\odot} = (\hat{x}_o^{\oplus} \cap \hat{x}_i^{\oplus}) \cap (\hat{x}_o^{\odot} \cap \hat{x}_i^{\odot})$. Using previous equality, it is easy to notice that, if at least one of the hypergroupoids $(R, \oplus)$ or $(R, \odot)$ is reduced, then the hyperring $(R, \oplus, \odot)$ is reduced, too. Reciprocally, if $(R, \oplus, \odot)$ is reduced, then the hypergroupoids $(R, \oplus)$ and $(R, \odot)$ can be reduced or not, which will be confirmed by the following examples.

Example 4.1. Let $(R, \oplus, \odot)$ be a hyperring defined by the following Cayley tables:

$\oplus$	e	a
e	R	R
a	R	R

$\odot$	e	a
e	e	R
a	R	a

Since $(R, \oplus)$ is a total hypergroup, based on Example 3.3, it is not reduced. Here, $\hat{a}_e^{\oplus} = \hat{e}_e^{\oplus} = \{e, a\}$. However, it is elementary to check that the hypergroup $(R, \odot)$ is a reduced hypergroup, and $\hat{a}_e^{\odot} = \{a\}, \hat{e}_e^{\odot} = \{e\}$. All together, it gives that $\hat{e}_e = \{e\}$ and $\hat{a}_e = \{a\}$ which gives that $(R, \oplus, \odot)$ is a reduced hyperring.

Example 4.2. Let the hyperring $(R, \oplus, \odot)$ be a hyperring defined by the next Cayley tables:

$\oplus$	x	y	z
x	x, y	x, y	R
y	x, y	x, y	R
z	R	R	R

$\odot$	x	y	z
x	R	R	R
y	R	y, z	y, z
z	R	y, z	y, z

It is elementary to check that the algebraic hyperstructure $(R, \oplus, \odot)$ is a general hyperring. Since the rows corresponding to x and y are equal in $(R, \oplus)$ and they appear in the same hyperproducts $a \oplus b$, it follows that $x \sim_e^{\oplus} y$, which implies that $(R, \oplus)$ is not reduced. Similarly, $(R, \odot)$ is not reduced hypergroup since $y \sim_e^{\odot} z$. But, $\hat{x}_e = \hat{x}_e^{\oplus} \cap \hat{x}_e^{\odot} = \{x, y\} \cap x = \{x\}$. Similarly, $\hat{y}_e = \{y\}$, and $\hat{z}_e = \{z\}$, which proves that $(R, \oplus, \odot)$ is a reduced hyperring.

Proposition 4.1. *Any subhyperring $(K, +, \cdot)$ of a reduced hyperring $(R, +, \cdot)$ is reduced, as well.*

Proof. The result easily follows from the law of contradiction. □

Remark 4.1. *A subhyperring of a non-reduced hyperring can be reduced or not. As we can see in Example 1.14, the hyperfield (which is a hyperring, too) $(H, \oplus, \odot)$ is not reduced, but its hyperideal (hence the subhyperring) is a reduced hyperring.*

4.1.1 Some properties of the reducibility in hyperrings

In the following subsections we suppose that the ring $(R, +, \cdot)$ have no zero divisors.
As we pointed out before, by a hyperring we mean a general hyperring.
Let us first present certain relationships between equivalence relations in particular hyperrings.

Proposition 4.2. *Let $(R, \oplus, \odot)$ be a general hyperring, where the hypergroup $(R, \oplus)$ contains a scalar identity. Then the essential indistinguishability with respect to the hyperoperation "$\oplus$" implies the essential indistinguishability with respect to the hyperoperation "$\odot$", i.e., $x \sim_e^{\oplus} y \Rightarrow x \sim_e^{\odot} y$, for all $x, y \in R$.*

Proof. We denote by 0 the scalar identity in $(R, \oplus)$. Let x and y be two elements in R such that $x \sim_o^{\oplus} y$, i.e., $x \oplus a = y \oplus a$ and $a \oplus x = a \oplus y$, for all $a \in R$. This means that, for any $u \in R$ such that $u \in x \oplus a$, it holds $u \in y \oplus a$. Let u belong to $a \odot x$. Then, since $x = x \oplus 0$, it follows that $u \in a \odot (x \oplus 0)$. Now, using $x \oplus 0 = y \oplus 0$, we get $u \in a \odot (y \oplus 0) = a \odot y$. By symmetry, we can conclude that $a \odot x = a \odot y$, and $x \odot a = y \odot a$, for all $a \in R$. Hence, $x \sim_o^{\odot} y$.

Let us suppose that $x \in a \oplus b$ if and only if $y \in a \oplus b$, for any $a, b \in R$. Let c and d be elements in the hyperring such that $x \in c \odot d$. Thus, $x \in (c \oplus 0) \odot d$. Using the distributibivity, we obtain $x \in c \odot d \oplus 0 \odot d = \{m \oplus n | m \in c \odot d, n \in 0 \odot d\}$. Since x and y appear in the same hyperproducts $a \oplus b$, for any $a, b \in R$, it follows that y also belongs to the same hyperproduct, which gives $y \in c \odot d \oplus 0 \odot d$, i.e., $y \in c \odot d$. This proves the implication $x \sim_i^{\oplus} y \Rightarrow x \sim_i^{\odot} y$. Now the conclusion of the result is clear. □

Corollary 4.1. *Let $(R, \oplus, \odot)$ be a general hyperring such that $(R, \oplus)$ contains a scalar identity. If $(R, \oplus)$ is not a reduced hypergroup, then the hyperring $(R, \oplus, \odot)$ is not reduced, too.*

Proof. If $(R, \oplus)$ is not a reduced hypergroup, then there exist two distinct elements x and y in R such that $x \sim_e^{\odot} y$, meaning that the hyperring $(R, \oplus, \odot)$ is not reduced. □

Let us now consider the hyperrings of formal series, where the reducibility is strictly connected with the reducibility of the general hyperring of coefficients.

Proposition 4.3. *Let $R[[x]]$ be the hyperring of the formal series with coefficients in the general commutative hyperring $(R, +, \cdot)$. The hyperring $(R, +, \cdot)$ is reduced if and only if the hyperring $(R[[x]], \oplus, \odot)$ is reduced.*

Proof. Let us suppose that the hyperring R is not reduced, i.e., there exist elements a and b such that $a + x = b + x$ and $x + a = x + b$ for all $x \in R$ and also a and b appear in the same hyperproducts $c + d$, where $c, d \in R$. As a direct consequence, the formal series $(a, a, \ldots, a, \ldots)$ and $(b, b, \ldots, b, \ldots)$ are operationally equivalent and inseparable with respect to the hyperoperation $\oplus$. Analogously, the implication holds also if we consider the multiplicative hyperoperation. Hence, if R is not reduced, then the hyperring $(R[[x]], \oplus, \odot)$ is not reduced, too.

Let us prove now that the reducibility in $(R, +, \cdot)$ implies the reducibility in $(R[[x]], \oplus, \odot)$. For that purpose, let us assume that the hyperring $R[[x]]$ is not reduced. Then, there exist two formal series $(a_1, a_2, \ldots, a_n, \ldots)$ and $(b_1, b_2, \ldots, b_n, \ldots)$ which are operationally equivalent with respect to the hyperoperation $\oplus$. This implies that:

$$(a_1, a_2, \ldots, a_n, \ldots) \oplus (x_1, x_2, \ldots, x_n, \ldots) = \tag{4.1}$$
$$(b_1, b_2, \ldots, b_n, \ldots) \oplus (x_1, x_2, \ldots, x_n, \ldots), \tag{4.2}$$

and

$$(x_1, x_2, \ldots, x_n, \ldots) + (a_1, a_2, \ldots, a_n, \ldots) = \tag{4.3}$$
$$(x_1, x_2, \ldots, x_n, \ldots) + (b_1, b_2, \ldots, b_n, \ldots), \tag{4.4}$$

for any formal series $(x_1, x_2, \ldots, x_n, \ldots) \in R[[x]]$. Using the definition of the hyperaddition in $(R[[x]], \oplus, \odot)$, the previous equalities give that $a_i + x_i = b_i + x_i$ and $x_i + a_i = x_i + b_i$

for any arbitrary $x_i \in R$. Hence, $a_i \sim_o^+ b_i$ for any elements $a_i, b_i \in R$, which are the coordinates of the considered formal series. Assuming now that the series $(a_1, a_2, \ldots, a_n, \ldots)$ and $(b_1, b_2, \ldots, b_n, \ldots)$ are inseparable with respect to the hyperoperation $\oplus$, it easily follows that a_i and b_i appear in the same hyperproducts $c + d$, where $c, d \in R$, so they are inseparable with respect to the hyperproduct $+$ on R. Similarly we can prove that the essential indistinguishability with respect to hypermultiplication $\odot$ implies essential indistinguishability with respect to the hyperoperation $\cdot$. We finally get that $(R, +, \cdot)$ is not reduced, which concludes the proof. □

The next part of this subsection is dedicated to the study of reducibility of hyperrings with P-hyperoperations. Here, we consider rings which are integral domains and whose ideals are principal, i.e., generated by a single element.

Proposition 4.4. *Let $(R, +, \cdot)$ be a commutative principal ideal domain with two units, i.e., 1 and -1. If $P_1 = nR$, with $n \in R$, and $P_2 = R$, then the structure (R, P_1^*, P_2^*) is a commutative H_v-ring with P-hyperoperations, which is a non-reduced hyperring.*

Proof. Any principal ideal contains 0, therefore $0 \in P_1$. Because the ring R is commutative, it coincides with its center $Z(R)$, and therefore the set $P_2 = R$ has a non-empty intersection with $Z(R)$, so the conditions of Theorem 1.12 are satisfied, proving that the hyperstructure (R, P_1^*, P_2^*) is a commutative H_v-ring.

Let x and y be distinct elements such that $xP_1^*a = yP_1^*a$ for all a in R, meaning that $x + a + P_1 = y + a + P_1$, i.e., $x + a + nR = y + a + nR$, for the fixed element $n \in R$ and any $a \in R$. Since the principal ideal nR is a subgroup, then the equality holds whenever $x - y \in nR$. Therefore, the elements x and y are operationally equivalent with respect to the hyperoperation P_1^* if and only if $x - y \in nR$.

Let x and y be two elements such that $x - y \in nR$. Let us suppose that $x \in aP_1^*b$, where $a, b \in R$. The element x belongs to $a + b + nR$, i.e., $x = a + b + n \cdot s$, whith $s \in R$. Since $x = y + n \cdot k$, with $k \in R$, it follows that $y + n \cdot k = a + b + n \cdot s$, meaning that $y \in a + b + nR$. Hence, $y \in aP_1^*b$. Similarly we can prove the other implication. Thus, $x \sim_i^{P_1^*} y$. Conversely, if $x \sim_i^{P_1^*} y$, then it is clear that $x - y \in nR = P_1$. Hence, for any two distinct elements $x, y \in R$, $x \sim_e^{P_1^*} y$ if and only if $x - y \in P_1$.

Now, suppose that x and y are operationally equivalent with respect to the hyperoperation P_2^*. Thus $xP_2^*a = yP_2^*a$, i.e., $x \cdot a \cdot P_2 = y \cdot a \cdot P_2$, for any $a \in R$. Using the property that two principal ideals are equal when their generators are associated, we obtain that there exists a unit u such that $ya = uxa$, and similarly, there exists a unit v such that $xa = vya$. Both together imply that $ya = uvya$, with $uv = 1$. Since the ring R contains only two units, we have exactly two possibilities. If both units u and v are the multiplicative identity 1, then we obtain that $xa - ya = 0$, i.e., $(x-y)a = 0$ which implies that $x = y$. The second case is when $u = v = -1$ and we obtain $ya = -xa$, for any $a \in R$, thus $y = -x$.

Regarding the inseparability with respect to the hyperoperation P_2^*, we easily see that for any $x \in R$, there is $x \sim_i^{P_2^*} (-x)$ and moreover $x \sim_e^{P_2^*} (-x)$.

Based on these two results, it follows clearly that $x \sim_e (-x)$, for any $x \in P_1$ which says that the H_v-ring (R, P_1^*, P_2^*) is not reduced. □

Example 4.3. An example of an H_v-ring with P-hyperoperations satisfying Proposition 4.4 can be obtained taking $R = \mathbb{Z}$, the ring of integers. In this particular case, the elements nk and $-nk$ with $k \in \mathbb{Z}$ are essentially indistinguishable.

If we restrict the set P_2 to non-negative integers, we obtain a reduced hyperring, as we can see in the following example.

Example 4.4. Let $\mathbb{Z}$ be the ring of integers and set $P_1 = n\mathbb{Z}$ with $n \in \mathbb{Z}$ and $P_2 = \mathbb{Z}^+$, the set of positive integers. Then the hyperstructure $(\mathbb{Z}, P_1^*, P_2^*)$ is a commutative H_v-ring with P-hyperoperations which is reduced.

It is evident that the conditions of the Theorem 1.12 are all fulfilled, which implies that the hyperstructure $(\mathbb{Z}, P_1^*, P_2^*)$ is an H_v-ring. Similarly as in Example 4.3, we conclude that $x \sim_e^{P_1^*} y$ if and only if $x - y \in P_1$, i.e., $x - y = ns$ for some $s \in \mathbb{Z}$.

Let us suppose that $xP_2^*a = yP_2^*a$, i.e., $x \cdot a \cdot \mathbb{Z}^+ = y \cdot a \cdot \mathbb{Z}^+$, for any $a \in \mathbb{Z}$. Choosing $a = 1$, it follows that $\{xk \mid k \in \mathbb{Z}^+\} = \{yk \mid k \in \mathbb{Z}^+\}$. The equality is satisfied only in the case when $x = y$. Thus, the H_v-ring $(\mathbb{Z}, P_1^*, P_2^*)$ is reduced.

Example 4.5. Let (R, P_1^*, P_2^*) be a commutative H_v- ring with $P-$ hyperoperations such that $(R, \cdot)$ is a group and let P_1 be a subgroup of $(R, +)$ and $P_2 = R$. Then the H_v-ring (R, P_1^*, P_2^*) is not reduced.

It is easy to check that the hyperstructure (R, P_1^*, P_2^*) is an H_v-ring with P-hyperoperations. Let us prove its non-reducibility. Indeed, following the procedure explained in Proposition 4.4, we conclude that $x \sim_e^{P_1^*} y$ if and only if $x - y \in P_1$. Hence, for any two distinct elements $x, y \in R$, such that $x - y \in P_1$, there is $\hat{x}_e^{P_1^*} = \hat{y}_e^{P_1^*} \supseteq \{x, y\}$. Taking $P_2 = R$ we easily get that $xP_2^*a = yP_2^*a$, for all $a \in R$, and if x belongs to aP_2^*b, obviously also y belongs to it. Therefore, for an arbitrary element x in R, there is $\hat{x}_e^{P_2^*} = R$.

Combining the two results, we get $x \sim_e y$, whenever $x - y \in P_1$, meaning that the considered H_v-ring is not reduced.

Example 4.6. Let (R, P_1^*, P_2^*) be a commutative H_v-ring with P-hyperoperations, such that $(R, +, \cdot)$ is a field and let K be a subfield of R. If $P_1 = P_2 = K$, then the H_v- ring (R, P_1^*, P_2^*) is not reduced.

Let x and y be arbitrary elements from R. Analogously to Example 4.5, $x \sim_e^{P_1^*} y$ if and only if $x - y \in P_1$.

Let us suppose that the equality $xP_2^*a = yP_2^*a$ is satisfied for all $a \in R$, i.e., $xaK = yaK$ for any $a \in R$. This is equivalent to $xK = yK$, which is satisfied for any $x, y \in K$. It is obvious that $x \sim_e^{P_1^*} y$, for arbitrary $x, y \in K$.

Merging both conclusions, we get that the hyperring (R, P_1^*, P_2^*) is not reduced, since any two elements x and y in K are essentially indistinguishable.

In the latest part of this subsection, we consider the reducibility in hyperrings containing Corsini hypergroups.

Proposition 4.5. *Let $(H, \circ)$ be a Corsini hypergroup and $(H, \star)$ be a B-hypergroup, i.e., $x \star y = \{x, y\}$ for all $x, y \in H$. Then the hyperring $(H, \star, \circ)$ is a reduced hyperring.*

Proof. Based on Al-Tahan and Davvaz [2], it is known that, if $(H, \circ)$ is a Corsini hypergroup and $(H, \star)$ is a B-hypergroup, then the structure $(H, \star, \circ)$ is a commutative hyperring. Kankaras has proved in [44] that any $B-$ hypergroup is a reduced hypergroup, which easily gives that the hyperring $(H, \star, \circ)$ is reduced, too. $\square$

Example 4.7. Endow the set $R = \{x, y, z\}$ with the hyperoperations $\oplus$ and $\odot$ given by the following tables:

$\oplus$	x	y	z
x	x, y	x, y	R
y	x, y	x, y	R
z	R	R	z

$\odot$	x	y	z
x	x	x, y	x, z
y	x, y	y	y, z
z	x, z	y, z	z

The hypergroup $(R, \oplus)$ is a Corsini hypergroup [2] and $(R, \odot)$ is a B-hypergroup. Here, $x \oplus a = y \oplus a$ for any $a \in R$. Thus, $x \sim_o^{\oplus} y$. Also, x and y appear in the same hyperproducts, which gives $x \sim_i^{\oplus} y$. Considering the second hyperoperation, it easily follows that $\hat{x}_e^{\odot} = \{x\}$ for any $x \in R$. Hence, $(R, \oplus, \odot)$ is a reduced hyperring.

Remark 4.2. If we take that $(R, \oplus)$ is the hypergroup defined in the Example 4.7 and $(R, \odot)$ be a total hypergroup, then both hypergroups are Corsini hypergroups, but the hyperring $(R, \oplus, \odot)$ is not a reduced hyperring since $\hat{x}_e = \hat{y}_e = \{x, y\}$.

Remark 4.3. Let $(R,\oplus)$ be an arbitrary hypergroup and $(R,\odot)$ be a total hypergroup. The algebraic hyperstructure $(R,\oplus,\odot)$ is a general hyperring and its reducibility depends on the reducibility of a $(R,\oplus)$. It $(R,\oplus)$ is a reduced hypergroup, then the hyperring $(R,\oplus,\odot)$ is reduced, too, otherwise, $(R,\oplus,\odot)$ is not reduced.

4.1.2 Reducibility in complete hyperrings

Before we introduce the definition of the complete hyperring, let us first recall the procedure of the construction of a complete hypergroup.

Theorem 4.1. [19] *Any complete hypergroup may be formed as the union* $H=\bigcup\limits_{g\in G} A_g$ *of its subsets, where*

1) $(G,\cdot)$ *is a group.*
2) *The family* $\{A_g,|g\in G\}$ *is a partition of* G, *i.e. for any* $(g_1,g_2)\in G^2$, $g_1\neq g_2$, *there is* $A_{g_1}\cap A_{g_2}=\emptyset$.
3) *If* $(a,b)\in A_{g_1}\times A_{g_2}$, *then* $a\circ b=A_{g_1g_2}$.

Definition 4.3. [34] *Let* $(H,\oplus,\odot)$ *be a hyperring. If* $(H,\oplus)$ *is a complete hypergroup, then we say that* H *is* $\oplus-$ *complete. If* $(H,\odot)$ *is a complete semihypergroup, then we say that* H *is* $\odot-$*complete and if both* $(H,\oplus)$ *and* $(H,\odot)$ *are complete, then we say that* H *is a complete hyperring.*

Based on the construction of complete hypergroups, De Salvo [34] proposed a method to obtain complete hyperring starting with rings. Let us recall here this construction:

Let $(R,+,\cdot)$ be a ring, and $\{A(g)\}_{g\in R}$ be a family of non-empty sets, such that:

1. $\forall g,g^{'}\in R, g\neq g^{'}\Rightarrow A(g)\cap A(g^{'})=\emptyset$
2. $g\notin R\cdot R\Rightarrow |A(g)|=1.$

Set $H_R=\bigcup_{g\in R}A(g)$ and define two hyperoperations $\oplus$ and $\odot$ on H_R in the following way:
For any $a,b\in H_R$, there exist $g,g^{'}\in R$ such that $a\in A(g), b\in A(g^{'})$. Then set

$$a\oplus b=A(g+g^{'}), a\odot b=A(gg^{'}).$$

Moreover, it was proved that:

Lemma 4.1. [34] *For all $g, g' \in R, \forall u \in A(g), \forall v \in A(g')$ we have:*
$u \oplus v = A(g + g') = A(g) \oplus A(g')$
$u \odot v = A(gg') = A(g) \odot A(g')$

In [9] Corsini proved that $(H_R, \oplus)$ and $(H_R, \odot)$ are, respectively, a complete commutative hypergroup and a complete semihypergroup.

Remark 4.4. All complete hyperrings can be constructed by the above described procedure, since it is known that any complete semihypergroup (hypergroup) can be constructed as the union of disjunct sets $A(g), g \in G$.[13]

Theorem 4.2. [34] *H_R is a complete hyperring.*

Remark 4.5. The hyperstructure H_R is a general hyperring in the sense of Definition 1.24, which is also complete.

Notice than any complete (semi)hypergroup is not reduced, but, as we showed in Example 4.2, the non-reducibility of both, multiplicative and additive part of a hyperring does not imply the non-reducibility of a hyperring. For that reason, we need to prove that if two elements x and y from the hyperring R are operationally equivalent (inseparable) with respect to the hyperoperation $\oplus$, the same elements are operationally equivalent (inseparability) with respect to the hyperoperation $\odot$.

Theorem 4.3. *Any complete hyperring $(H_R, \oplus, \odot)$ is not reduced.*

Proof. Let $(H_R, \oplus, \odot)$ be a complete hyperring. Thereby the hypergroup $(H_R, \oplus)$ and the semihypergroup $(H_R, \odot)$ are both complete, so both are not reduced. Thence, we conclude that there exist $a \neq b \in H_R$ such that $a \sim_e^{\oplus} b$. Now it is enough to prove that $a \sim_e^{\oplus} b$ implies $a \sim_e^{\odot} b$ for $a, b \in H_R$, because in this case $\hat{a}_e = \hat{a}_e^{\oplus} \cap \hat{a}_e^{\oplus} \supseteq \{a, b\}$, which shows that $(H_R, \oplus, \odot)$ is not reduced.

First we will prove that the operational equivalence relation with respect to the hyperoperation $\oplus$ implies the operational equivalence relation with respect to $\odot$. Let a, b be elements from H_R such that $a \oplus c = b \oplus c$, for all $c \in H_R$. It follows that there exist $g_a, g_b, g_c \in R$ such that $a \in A(g_a), b \in A(g_b)$ and $c \in A(g_c)$. According to Lemma 4.1, we have $a \oplus c = A(g_a + g_c)$ and $b \oplus c = A(g_b + g_c)$, which leads to the equality $A(g_a + g_c) = A(g_b + g_c)$, equivalently with $g_a + g_c = g_b + g_c$ in the group $(R, +)$. Therefore $g_a = g_b$, that implies that $g_a \cdot g_c = g_b \cdot g_c$. Therefore, $a \odot c = A(g_a \cdot g_c) =$

$A(g_b \cdot g_c) = A(g_b) \odot A(g_c) = b \odot c$. Similarly, $c \oplus a = c \oplus b$ implies that $c \odot a = c \odot b$. This means that $a \sim_o^{\oplus} b$ implies $a \sim_o^{\odot} b$ for all $a, b \in H_R$.

Next we will show that the indistinguishability relation with respect to $\oplus$ implies the indistinguishability relation with respect to $\odot$.

Let us suppose $a \sim_i^{\oplus} b$. This means that a and b appear in the same hyperproducts $d \oplus e$, for $d, e \in H_R$. Thus $a \in A(g_d) \oplus A(g_e) \iff b \in A(g_d) \oplus A(g_e)$, with $g_d, g_e \in R$ such that $d \in A(g_d), e \in A(g_e)$. It follows that $a \in A(g_d + g_e) \iff b \in A(g_d + g_e)$, meaning that $a, b \in A(g)$, with $g \in R$. If we consider now $a \in k \odot l$, then $a \in A(g_k \cdot g_l)$, where $k \in A(g_k), l \in A(g_l)$. Since a and b are in the same A_g, it follows that $b \in A(g_k \cdot g_l) = k \odot l$, equivalently, $b \in k \odot l$. Similarly, if $b \in k \odot l$ then $a \in k \odot l$. Hence, $a \sim_i^{\odot} b$. □

Example 4.8. Let the hyperring $H = (\{a, b, c, d, e\}, \oplus, \odot)$ is defined as follows:

$\oplus$	a	b	c	d	e
a	a	b, c	b, c	d	e
b	b, c	d	d	e	a
c	b, c	d	d	e	a
d	d	e	e	a	b, c
e	e	a	a	b, c	d

$\odot$	a	b	c	d	e
a	a	a	a	a	a
b	a	b, c	b, c	d	e
c	a	b, c	b, c	d	e
d	a	d	d	a	d
e	a	e	e	d	b, c

The hyperring $(H, \oplus, \odot)$ is a commutative complete hyperring [32]. Since the rows corresponding to the elements b and c are exactly the same in both tables, we conclude that $b \sim_o^{\oplus} c$ and $b \sim_o^{\odot} c$, which further gives $b \sim_o c$, i.e., $\hat{b}_o = \hat{c}_o = \{b, c\}$. Besides, elements b and c appear together in $(H, \oplus)$, as well as in $(H, \odot)$, whence there is $b \sim_i c$, thus $\hat{b}_i = \hat{c}_i = \{b, c\}$. Hence, $\hat{b}_e = \hat{c}_e = \{b, c\}$ which implies that the given hyperring is not reduced.

4.1.3 Reducibility in $(H, R)-$ hyperrings

$(H, R)-$ hyperrings were introduced by Mario de Salvo in [34], with the intention of generalizing the construction of $(H, G)-$ hypergroups described in [33]. In the following we present their construction.

Let $(H, \circ, \Box)$ be a hyperring and $\{A_i\}_{i \in R}$ be a family of non-empty sets such that:

1. $(R, +, \cdot)$ is a ring.
2. $A_{0_R} = H$.

3. For any $i \neq j \in R, A_i \cap A_j = \emptyset$.

Set $K = \bigcup_{i \in R} A_i$ and define on set K hyperoperations as:

$$\text{for any} \quad x, y \in H, x \oplus y = x \circ y \tag{4.5}$$

$$\text{and} \quad x \odot y = H \tag{4.6}$$

For any $x \in A_i$ and $y \in A_j$ such that $A_i \times A_j \neq H \times H$, define

$$x \oplus y = A_k \quad \text{if} \quad i + j = k, \tag{4.7}$$

$$x \odot y = A_m \quad \text{if} \quad i \cdot j = m. \tag{4.8}$$

The structure $(K, \oplus, \odot)$ is a general hyperring, called an (H, R)-*hyperring*. Moreover, if ω is the heart of the hypergroup $(K, \oplus)$, then $\omega = H$ and $H \odot K = K \odot H = K$ [34].

In the following we will better describe the operational equivalence and the inseparability in an (H, R)-hyperring.

Lemma 4.2. *Let $(K, \oplus, \odot)$ be an (H, R)-hyperring, where $K = \bigcup_{i \in R} A_i$, with $(R, +, \cdot)$ a ring and $(H, \circ, \Box)$ a hyperring.*

1. *Two elements x and y in $A_{0_R} = H$ are operationally equivalent with respect to the hyperoperation $\oplus$ if and only if they are operationally equivalent with respect to the hyperoperation $\circ$ on H.*

2. *Two elements x and y in $K \setminus A_{0_R}$ are operationally equivalent with respect to the hyperoperation $\oplus$ if and only if they belong to the same subset $A_i \subset K$.*

3. *Two elements x and y in K are inseparable with respect to the hyperoperation $\oplus$ if and only if they belong to the same subset $A_i \subset K$.*

Proof. 1. Let x, y be in $A_{0_R} = H$ such that $x \oplus a = y \oplus a$, for all $a \in K$. If $a \in A_{i_a}$, with $i_a \neq 0_R$, then the equality always holds. If $a \in A_{0_R}$, then $x \oplus a = y \oplus a$ whenever $x \circ a = y \circ a$ and thus the result is proved.

2. Let x and y be in $K \setminus H$, such that $x \in A_{i_x}$ and $y \in A_{i_y}$, with $i_x, i_y \in R$ and consider $x \oplus a = y \oplus a$, for all $a \in K$. If $a \in A_{0_R}$, then $x \oplus a = A_{i_x}$ and $y \oplus a = A_{i_y}$, therefore x and y are operationally equivalent if and only if $i_x = i_y$. If $a \in K \setminus A_{0_R}$, for example $a \in A_{i_a}$, then $x \oplus a = y \oplus a$ is equivalent with $i_x + i_a = i_y + i_a$, meaning again $i_x = i_y$.

3. Let us consider $x \sim_i^{\oplus} y$, which is to say $x \in a \oplus b$ if and only if $y \in a \oplus b$. If $a, b \in A_{0_R}$, then $a \oplus b = a \circ b$ and therefore $x \sim_i^{\oplus} y$ whenever $x, y \in a \circ b \subset A_{0_R}$. If $a \in A_{i_a}$ and $b \in A_{i_b}$ with $A_{i_a} \times A_{i_b} \neq H \times H$, then $a \oplus b = A_{i_a + i_b} = A_i$ and therefore $x \sim_i^{\oplus} y$ whenever $x, y \in A_i$, with $i \in R$. Combining the two cases, we get that x and y are inseparable if and only if they are in the same subset A_i.

□

Lemma 4.3. *Let $(K, \oplus, \odot)$ be an (H, R)-hyperring, where $K = \bigcup_{i \in R} A_i$, with $(R, +, \cdot)$ an integral domain and $(H, \circ, \Box)$ a hyperring. Two elements x and y in K are essentially indistinguishable with respect to the hyperoperation $\odot$ if and only if they belong to the same subset $A_i \subset K$.*

Proof. The proof is similar to the one of Lemma 4.2. The only difference here is in the case of the relation "$\sim_o$", where the condition regarding R to be an integral domain is fundamental. □

Proposition 4.6. *Let $(K, \oplus, \odot)$ be an (H, R)-hyperring, where $K = \bigcup_{i \in R} A_i$, with $(R, +, \cdot)$ an integral domain and $(H, \circ, \Box)$ a hyperring. Then the hyperring $(K, \oplus, \odot)$ is not reduced if and only if there exists $i \in R, i \neq 0_R$, with $|A_i| \geq 2$, or the hypergroup $(H, \circ)$ is not reduced.*

Proof. Let us suppose that the hyperring $(K, \oplus, \odot)$ is not reduced. Then there exist two elements x and y in K such that $x \sim_e y$, i.e., $x \sim_e^{\oplus} y$ and $x \sim_e^{\odot} y$. Based on Lemma 4.2 and and Lemma 4.3, if x and y belong to the same subset A_i, with $i \neq 0_R$, we conclude that $|A_i| \geq 2$. Otherwise, if all sets $A_i, i \neq 0_R$ are singletons, then $x, y \in A_{0_R} = H$, which implies that $x \sim_o^{\circ} y$ and $x \sim_i^{\circ} y$, i.e., the structure $(H, \circ)$ is a not reduced hypergroup.

Conversely, suppose there exists $i \in R \setminus \{0_R\}$ such that $|A_i| \geq 2$. Then there exist two elements x and y in the set A_i, implying that $x \sim_e^{\oplus} y$ and $x \sim_e^{\odot} y$. In other words, $x \sim_e y$, signifying that the (H, R)-hyperring $(K, \oplus, \odot)$ is not reduced. Assuming that $(H, \circ)$ is not reduced, let x and y be two elements such that $x \sim_e^{\circ} y$. According with Lemma 4.2 and and Lemma 4.3, we further conclude that $x \sim_e^{\oplus} y$. Due to the definition of the hyperoperation $\odot$, for any $x, y \in H$, it easily follows that $x \sim_e^{\odot} y$. Hence, $x \sim_e y$, i.e., $(K, \oplus, \odot)$ is not a reduced hyperring. □

Corollary 4.2. *If $(H, \oplus, \odot)$ is a not reduced hyperring, then the (H, R)-hyperring $(K, \oplus, \odot)$ is not reduced, too.*

In the following, we will give an example of an (H, R)-hyperring and show its non-reducibility.

Example 4.9. *Let endow the set $R = \{0, a, b, c\}$ with the following operations*

$+$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	0	a	a
b	0	0	b	b
c	0	0	c	c

It easily follows that $(R, +, \cdot)$ is a ring. Furthermore, let $(H, \circ, \square)$ be a hyperring given by the tables

$\circ$	c	d
c	c	c, d
d	c, d	c, d

$\square$	c	d
c	c	c, d
d	c	c, d

The structure $(H, \circ, \square)$ is a general hyperring [34]. It is easy to check that $(H, \circ)$ is a reduced hypergroup and thus, the hyperring $(H, \circ, \square)$ is reduced, too.

We will endow the set $K = \{c, d, a_1, a_2, a_3, a_4, a_5, a_6\}$, where $A_0 = H$, $A_a = \{a_1, a_2\}$, $A_b = \{a_3, a_4, a_5\}$, $A_c = \{a_6\}$, with an (H, R)-hyperstructure by defining the hyperaddition $x \oplus y = x \circ y$ if both x, y belong to H, otherwise, let $x \oplus y = A_k$, with $x \in A_i, y \in A_j$ and $k = i + j$. Besides, define $x \odot y = H$, where $x, y \in H$ and $x \oplus y = A_k$ with $x \in A_i, y \in A_j$ and $k = i \cdot j$. Then the structure $(K, \oplus, \odot)$ is an (H, R)-hyperring.

Let us prove that $a_1 \sim_e a_2$, i.e., $a_1 \oplus x = a_2 \oplus x$ for all $x \in K$. Indeed, if $x \in H, a_1 \oplus x = A_{a+0} = a_2 \oplus x$. If $x \in A_a, a_1 \oplus x = A_{a+a} = A_0 = a_2 \oplus x = A_0$. For $x \in A_b, a_1 \oplus x = A_{a+b} = A_c = a_2 \oplus x$. Finally, $a_1 \oplus x = a_2 \oplus x = A_{a+c} = A_b$ for $x \in A_c$. Due to the commutativity of the ring R, $x \oplus a_1 = x \oplus a_2$ for any $x \in K$. Similarly, $a_1 \odot x = a_2 \odot x$ and $x \odot a_1 = x \odot a_2$ for any $x \in K$. Thus, $a_1 \sim_o a_2$.

Since $x \oplus y \subset H$ if both $x, y \in H = A_0 = \{c, d\}$, we conclude that the elements a_1 and a_2 don't appear in such hyperproducts. All other hyperproducts $x \oplus y$ are equal to some sets A_k, where $k \in \{a, b, c\}$, with A_a, A_b and A_c being disjoint sets. Hence, a_1 and a_2 appear in the hyperproducts which are equal to A_a, so they always appear together. Analogously, a_1 and a_2 appear in the same hyperproducts $x \odot y$. Hence, $a_1 \sim_i a_2$.

Similarly one proves that $\hat{a}_{3e} = \hat{a}_{4e} = \hat{a}_{5e} = \{a_3, a_4, a_5\}$. Thereby we conclude that the $(H, R)-$ hyperring $(K, \oplus, \odot)$ is not reduced.

Chapter 5

Conclusions

This thesis deals with the study of reducibility in different algebraic hyperstructures. The motivation for our study was given by Jantosciak, who noticed that in a hypergroup elements can play the same role with respect to the hyperoperation. He defined certain equivalence relations which are the basis for the definition of reducibility. The equivalence relations are called: the operational equivalence, inseparability and essential indistinguishability. Jantosciak gave the definition of a reduced hypergroup, saying that the hypergroup is reduced if the equivalence class of each element with respect to the relation "essential indistinguishability" is one-element set. In the dissertation, we have extended the classical notion of reducibility to the fuzzy case, by defining new concepts: *fuzzy reduced hypergroup* and *reduced fuzzy hypergroup*. We can consider the reducibility on a crisp hypergroup equipped with a fuzzy set or in a fuzzy hypergroup. In the first case we examine a fuzzy reduced hypergroup, i.e., crisp hypergroup that is fuzzy reduced with respect to the corresponding fuzzy set. In the second case, using fuzzy hyperproducts, we get reduced fuzzy hypergroup, in other words fuzzy hypergroup that is reduced. We have developed the first case and studied the fuzzy reducibility in hypergroups. In order to extend the concept of the reducibility to the fuzzy case, we have introduced, similarly to the crisp case, equivalence relations called: the fuzzy operationally equivalence, the fuzzy inseparability and the fuzzy essentially indistinguishability, taking into consideration the fact that now we are working with fuzzy sets. A crisp hypergroup equipped with a fuzzy set is called a fuzzy reduced hypergroup if and only if each element in the hypergroup has singleton equivalence class with respect to the relation fuzzy essential indistinguishability.

Moreover, we have extended the reducibility concept to hyperrings, by defining equivalence relations with respect to the both hyperoperations in a general hyperring. Analogously to the hypergroups, we have introduced the definition of a reduced hyperring.

In this Phd thesis we have investigated the reducibility (in both, crisp and fuzzy case) for many classes of hypergroups and the reducibility in hyperrings. After the preliminaries chapter which contains the basic hyperstructure theory necessary for our study, in the second chapter we have presented our results related to the reducibility in hypergroups. In the third chapter we have studied the fuzzy case of the reducibility, i.e., we have studied the fuzzy reducibility in hypergroups. The fourth chapter deals with a hyperring reducibility. After we have extended the concept of the reducibility to the hyperrings (in the crisp case), we have investigated the reducibility property in certain classes of general hyperrings.

The main results obtained in the dissertation are related to the hypergroups reducibility. Regarding the crisp case, we have proved that complete hypergroups are not reduced, while the hypergroups with partial scalar identities are reduced. Moreover, arbitrary canonical hypergroups is a reduced hypergroup. Also, we have found the necessary and sufficient condition for Corsini hypergroups to be reduced. We have concluded that the reducibility of cyclic hypergroups strictly depends of their cardinality and of the number of generators, too. When it comes to the fuzzy reducibility, the complete hypergroups are not reduced, as well as in the crisp case, while i.p.s. hypergroups are not fuzzy reduced. We have proved that Corsini hypergroups are not fuzzy reduced. We have to emphasize that in all cases the fuzzy reducibility has been investigated with respect to the grade fuzzy set $\widetilde{\mu}$. As for the hyperrings, the reducibility have been studied only in the crisp case. Using the result about non-reducibility of complete hypergroups, we have got the same conclusion for the complete hyperrings. Besides, we have determined conditions under which the $(H, R)-$ hyperring is reduced.

Many of the results presented here can be found in the articles *Fuzzy reduced hypergroups* and *Reducibility in Corsini hypergroups*, published in the journals "Mathematics" and "*Analele Stiintifice Universitatii Ovidius Constanta, Seria Matematica*", respectively.

The idea concerning our study, which will be the topic of our further research, is the second extension of the reducibility concept to the fuzzy case. Our goal is to examine the reducibility in fuzzy hyperstructures, especially in hypergroups. In the following we give the examples of some fuzzy hypergroups which are reduced.

Example 5.1. Let H be a non-empty set and $\circ : H \times H \to F(H)$, where $a \circ b = \chi_{\{a,b\}}$. $(H, \circ)$ is a reduced fuzzy hypergroup.
In [60] can be found the proof that the structure $(H, \circ)$ is a fuzzy hypergroup.
Let $a, b \in H$ be the elements such that $a \sim_o b$, i.e., $(a \circ x)(r) = (b \circ x)(r)$ for all $x, r \in H$.

If we take that $x = a$ then we obtain that $(a \circ a)(r) = (b \circ a)(r)$ for all $r \in H$. Since

$$(a \circ a)(r) = \chi_{\{a\}}(r) = \begin{cases} 1; r = a \\ 0; r \neq a \end{cases} \tag{5.1}$$

then $(a \circ a)(a) = 1$. But, $(a \circ a)(a) = (b \circ a)(a) = \chi_{\{b,a\}}(a)$, so $\chi_{\{b,a\}}(a) = 1$. Also, $(a \circ a)(b) = (b \circ a)(b) = \chi_{\{b,a\}}(b) = 1$, which implies that $\chi_a(b) = 1$, meaning that $b = a$. Thus, we obtain that $a \sim_o b$ if and only if $a = b$. Hence, for all $a \in H, \hat{a}_o = \{a\}$ and thereby $\hat{a}_e = \{a\}$ which gives that $(H, \circ)$ is a reduced fuzzy hypergroup.

Example 5.2. Let $(H, \cdot)$ be a group with identity e, and the hyperoperation $\circ : H \times H \to F(H)$ is given with $a \circ b = \chi_{ab}$. Then the hypergroup $(H, \circ)$ is a reduced fuzzy hypergroup.
Let us suppose that $a \sim_o b$, i.e., $(a \circ x)(r) = (b \circ x)(r)$ for all $x, r \in H$. By taking that $x = e$, we get $(a \circ e) = \chi_{ae} = \chi_a$. Now, $1 = (a \circ e)(a) = (b \circ e)(a) = \chi_b(a)$, thus $a = b$. Hence, $(H, \circ)$ is a reduced fuzzy hypergroup.

Based on the previous examples, we aim to conduct our study about the reducibility of fuzzy hyperstructures and to find some general results. Also, we intend to extend the reducibility in hyperrings to the fuzzy case, studying the fuzzy reducibility for the wide classes of general hyperrings.

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Biografy

Milica Kankaraš was born in Nikšić on 4 April 1988. In 2006, she graduated from the high school "Stojan Cerović" in Nikšić and enrolled in the Faculty of Mathematics and Natural Sciences at the University of Montenegro, in Podgorica. In 2009, she obtained B.Sc. degree in Mathematics and Natural Sciences. One year after, she obtained a Spec. Sci degree in Mathematics and Natural Sciences. In 2012, she became a master of sciences and recently after that she started with her Phd studies. She published the article "Fuzzy reduced hypergroups" in the journal "Mathematics" together with Irina Cristea in February, 2020. In March, 2021, she published the article "Reducibility in Corsini hypergroup" in the journal "Analele Stiintifice ale Universitatii Ovidius Constanta".

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